

On the geometry of string duals with backreacting flavors

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Abstract

Making use of generalized calibrated geometry and G -structures we put the problem of finding string-duals with smeared backreacting flavor branes in a more mathematical setting. This more formal treatment of the problem allows us to easily smear branes without good coordinate representations, establish constraints on the smearing form and identify a topological central charge in the SUSY algebra. After exhibiting our methods for a series of well known examples, we apply them to the problem of flavoring a supergravity-dual to a $d = 2 + 1$ dimensional $\mathcal{N} = 2$ super Yang-Mills-like theory. We find new solutions to both the flavored and unflavored systems. Interpretating these turns out to be difficult.

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1 Introduction

The AdS/CFT correspondence [1, 2] gives a by now well understood duality between type IIB string theory on $AdS_5 \times S^5$ and maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills. This high level of supersymmetry stands in stark contrast to the gauge theories of the standard model and so the correspondence was quickly extended to duals of gauge theories with reduced supersymmetry by either placing branes at singularities [3][4][5] or wrapping them on collapsing cycles [6][7][8][9][10].

In the search for something akin to a string dual of QCD, a further undesirable feature of the original AdS/CFT duality is the lack of matter fields charged under the fundamental representation of the gauge group. The issue, addressed in [11], was resolved by the inclusion of N_f probe flavor branes into the background. As these are taken to wrap non-compact cycles, their world-volume gauge theories become infinitely weakly coupled and non dynamical from a lower dimensional point of view. Therefore those branes provide the global flavor symmetry group for the gauge theory. One has to assume $N_f \ll N_c$ for the probe approximation to be valid.

However the study of some aspects of both QCD and supersymmetric gauge theories demands for a discussion of $N_f \sim N_c$, so it is imperative to go beyond the probe approximation. When doing so one should keep in mind a fundamental difference between the color and flavor branes. While the former undergo a geometric transition and are replaced by their fluxes, the latter are still present in the string dual – whether one includes their backreaction or treats them as

probes – in order to realize the $SU(N_f)$ flavor symmetry in the bulk. Therefore one needs to consider the combined action

$$S = S_{IIB} + S_{\text{Branes}} \quad (1.1)$$

where $S_{\text{Branes}} = S_{DBI} + S_{WZ}$ is the brane action given by the DBI and Wess-Zumino terms. This method of computing the backreaction was introduced in [12]. The localized branes become δ -function sources in the equations of motion, making the search for solutions of the above system highly non-trivial. See however [13, 14].

The most successful method of dealing with this issue was first developed in [15]. See also [16, 17, 18]. By considering a continuous distribution of flavor branes over their transverse directions one avoids the problem of the inclusion of localized sources, making the search for solutions much more feasible. An advantage of this so-called smearing method lies in the fact that the inclusion of localized sources breaks local isometries on the string theory side, leading to a violation of global symmetries in the full dual gauge theory – including the Kaluza-Klein modes. Such symmetries are restored after smearing. By now, many examples of string duals with both massless and massive flavors in various dimensions have been constructed [19][20][21][22][23][24][25][26][27][28][29][30].

Smearing of Dp -branes is usually determined by use of a $(10 - p - 1)$ -form Ω , the smearing form, which is in general interpreted as a distribution density of the branes. One issue resides in finding a way to construct Ω , such that it will be possible to find a solution to the equations of motion. This problem is usually dealt case by case and can be quite difficult to address when flavor embeddings cannot be identified with globally defined coordinates. If there is no obvious choice of Ω , one uses the physical properties of the anticipated dual gauge theory such as the mass of the fundamental fields or the unbroken symmetries, in order to impose constraints on its form.

In order to address this problem, we will make use of some of the methods of modern string phenomenology. If one allows for the misnomer of thinking of string duals with flavors as string compactifications with non-compact internal manifolds, the two fields become virtually the same. So it is quite striking that, while the set-up of gauge/string duality with flavors is quite similar to that of string phenomenology, the methods used are quite different. Especially the advanced mathematical methods of modern phenomenology such as the uses of generalized calibrated geometry, G -structures or generalized geometry, have up to now been absent from any discussion of flavors and duality.

Geometric arguments have been used to tackle the question of supersymmetry conservation, but not to answer the issue of smeared flavors. However geometry is one of the main techniques used in the study of string compactifications, which is in many points similar to the search for backgrounds with gauge duals.

So this paper is a first step towards bridging the gap between string phenomenology and gauge/string dualities with flavors, by presenting a systematic method of finding backgrounds with smeared flavors, using tools from modern geometry. The main goal is to find the smearing form Ω . The strategy is to use generalized calibrated geometry [31]. The central concept of this field is that of the calibration form $\hat{\phi}$, a $(p + 1)$ -form which can usually be constructed as a bilinear of the supersymmetry spinors of the background. It has the property

that a brane is supersymmetric if and only if the pull-back $X^*\hat{\phi}$ of the form onto the world-volume is equal to the induced volume form. It follows immediately that one can write the DBI action of any supersymmetric brane in terms of the pull-back of the calibration form. As all the backgrounds considered in this paper are type IIB backgrounds with only the dilaton Φ and one or two Ramond-Ramond fields excited, it will not be necessary to make use of the full machinery of generalized calibrations. The reader interested should refer to [32, 33, 34] and references therein.

Let us turn to the central argument of this paper. In the case of type IIA/B backgrounds with Ramond-Ramond flux $F_{(p+2)}$, we can write the action of the smeared flavor branes in Einstein frame as (see [34])

$$S_{\text{Branes}} = -T_p \int_{\mathcal{M}_{10}} (e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)}) \wedge \Omega \quad (1.2)$$

As we will see, it is always possible to relate the smearing form to the calibration form using supersymmetry and the equations of motion as

$$d[*e^{\frac{10-2p-4}{4}\Phi} d(e^{\frac{p-3}{4}\Phi} \hat{\phi})] = \pm 2\kappa_{10}^2 T_p \Omega \quad (1.3)$$

giving us a geometric constraint on the smearing form. In the following we shall study how equations (1.2) and (1.3) can be applied to address the problem of smeared flavors.

Proceeding rather pedagogically, section 2.1 introduces the methods outlined above studying three different, well-known examples. We will see that our methods are not only capable of reproducing the known results, yet also provide some new, interesting ones. The examples studied are the $\mathcal{N} = 1$ sQCD-like dual of [16, 17, 18], the $d = 2 + 1$ dimensional $\mathcal{N} = 1$ theory of [25] and the Klebanov-Witten theory [3] with massless [19] and massive [24] flavors. Following this we shall turn to the generic case (section 2.2), showing how the action (1.2) can be constructed from purely geometric considerations and proving its equivalence with other actions used in the field of smeared flavors.

In section 3 we shall finally apply our methods to the problem of flavoring a background dual to an $\mathcal{N} = 2$ super Yang-Mills-like theory first studied in [8, 36]. We will see that we are able to do so without an explicit knowledge of the brane embeddings used. We find new analytic and asymptotic solutions to the flavored and unflavored equations of motion and discuss various properties of these backgrounds.

Following [35] we will show for the examples considered, how all constraints imposed by supersymmetry upon space-time can be understood and recovered from geometric grounds using methods such as G -structures.

In the appendix A we give a short review of the required background in generalized calibrations. This is followed by a detailed example of how to calculate a calibration form in appendix B.

2 The geometry of smeared branes

In the following we shall now investigate what generalized calibrated geometry can teach us about string theory duals with backreacting, smeared flavor branes. First we will take a detailed look at three examples [16, 25, 19]. For each of these

we will briefly summarize the conventional approach to flavoring and will then show explicitly that it can be nicely understood in terms of a suitable calibration form. In section 2.2 we will turn to the case of a generic supergravity dual.

2.1 Three examples

2.1.1 The string dual to an $\mathcal{N} = 1$ sQCD-like theory

Review of the $\mathcal{N} = 1$ sQCD-like string dual As a first example we shall turn to the string dual to an $\mathcal{N} = 1$ sQCD-like theory [16, 17, 18]. It is based on the background of [7] which is given by the following solution of the type IIB equations of motion:¹

$$\begin{aligned} ds^2 &= \alpha' g_s N_c e^{\frac{\Phi}{2}} \left[\frac{1}{\alpha' g_s N_c} dx_{1,3}^2 + dr^2 + e^{2h} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{4} (\tilde{\omega}_i - A^i)^2 \right] \\ F_{(3)} &= -\frac{1}{4} \bigwedge_a (\tilde{\omega}_a - A^a) + \frac{1}{4} \sum_a F^a \wedge (\tilde{\omega}_a - A^a) \end{aligned} \quad (2.1)$$

with

$$\begin{aligned} A^1 &= -a(r) d\theta & \tilde{\omega}_1 &= \cos \psi d\tilde{\theta} + \sin \psi \sin \tilde{\theta} d\tilde{\phi} \\ A^2 &= a(r) \sin \theta d\phi & \tilde{\omega}_2 &= -\sin \psi d\tilde{\theta} + \cos \psi \sin \tilde{\theta} d\tilde{\phi} \\ A^3 &= -\cos \theta d\phi & \tilde{\omega}_3 &= d\psi + \cos \tilde{\theta} d\tilde{\phi} \end{aligned} \quad (2.2)$$

The metric describes a space with topology $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$, where the three-sphere is parametrized by the Maurer-Cartan forms $\tilde{\omega}_i$ and the one-forms A^i describe the fibration between the two spheres. It is interpreted as the near-horizon geometry of a stack of N_c D5-branes wrapping an S^2 , thus describing the dynamics of $d = 3 + 1$ dimensional $\mathcal{N} = 1, \mathcal{SU}(N_c)$ super Yang-Mills theory coupled to some extra matter. To keep the discussion as simple as possible, we shall focus on the so-called singular solution which is obtained from the assumption $a(r) = 0$.

The possibility of adding probe flavor branes to the above background (2.1) was studied in [37]. Using κ -symmetry the authors found several classes of flavor D5-branes; the simplest of these is given by branes extending along (x^μ, r) and wrapping ψ . They are pointlike on the four-dimensional submanifold given by $(\theta, \phi, \tilde{\theta}, \tilde{\phi})$ and extend to $r = 0$, thus describing massless flavors. In what follows, the most important feature of this embedding is that we are able to identify world-volume coordinates ξ^α with space-time ones, (x^μ, r, ψ) . So even at the level of the space-time coordinates X^M there is a very well defined notion of coordinates tangential and transverse to the brane.

From the perspective of type IIB string theory, it is clear that the addition of a large number of such branes to the system (2.1) will deform the geometry of the background. Given the form of the brane embeddings it follows that a

¹ Except where explicitly noted, we shall always use Einstein frame in this paper.

suitable ansatz for the deformed background should be of the form

$$\begin{aligned} ds^2 &= e^{2f(r)}[dx_{1,3}^2 + dr^2 + e^{2h(r)}(d\theta^2 + \sin^2 \theta d\phi^2) \\ &\quad + \frac{e^{2g(r)}}{4}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2) + \frac{e^{2k(r)}}{4}(\tilde{\omega}_3 + \cos \theta d\phi)^2] \\ F_{(3)} &= -2N_c e^{-3f-2g-k} e^{123} + \frac{N_c}{2} e^{-3f-2h-k} e^{\theta\phi 3} \end{aligned} \quad (2.3)$$

as the flavor branes are points on the four-dimensional transverse manifold while singling out the $\mathcal{U}(1) \subset S^3$ parametrized by ψ . When writing (2.3) we introduced a vielbein

$$\begin{aligned} e^{x^i} &= e^f dx^i & e^1 &= \frac{e^{f+g}}{2} \tilde{\omega}_1 & e^2 &= \frac{e^{f+g}}{2} \tilde{\omega}_2 & e^3 &= \frac{e^{f+k}}{2} (\tilde{\omega}_3 + \cos \theta d\phi) \\ e^r &= e^f dr & e^\theta &= e^{f+h} d\theta & e^\phi &= e^{f+h} \sin \theta d\phi \end{aligned} \quad (2.4)$$

and made use of the convention $e^{A_1 \dots A_p} = e^{A_1} \wedge \dots \wedge e^{A_p}$.

One can also interpret the ansatz (2.3) from the gauge theory point of view. The $\mathcal{U}(1)$ describes the R-symmetry of the flavored theory, which one demands not to be broken classically by the addition of massless flavors.

Studying the dilatino and gravitino variations of the deformed background one obtains the projections satisfied by the SUSY spinor ϵ ,

$$\Gamma_{r123}\epsilon = \epsilon \quad \Gamma_{r\theta\phi 3}\epsilon = \epsilon \quad \epsilon = \sigma_3 \epsilon \quad (2.5)$$

as well as the BPS equations

$$\begin{aligned} 4f &= \Phi \\ h' &= \frac{1}{4} N_c e^{-2h-k} + \frac{1}{4} e^{-2h+k} = \frac{1}{2} e^{3f} F_{\theta\phi 3} + \frac{1}{4} e^{-2h+k} \\ g' &= -N_c e^{-2g-k} + e^{-2g+k} = \frac{1}{2} e^{3f} F_{123} + e^{-2g+k} \\ k' &= \frac{1}{4} N_c e^{-2h-k} - N_c e^{-2g-k} - \frac{1}{4} e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\ &= \frac{1}{2} e^{3f} (F_{\theta\phi 3} + F_{123}) - \frac{1}{4} e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\ \Phi' &= -\frac{1}{4} N_c e^{-2h-k} + N_c e^{-2g-k} = -\frac{1}{2} e^{3f} (F_{\theta\phi 3} + F_{123}) \end{aligned} \quad (2.6)$$

It is a priori not obvious that the flavor branes mentioned earlier are still supersymmetric brane embeddings for the deformed background for arbitrary functions g, h, k . One therefore has to check again that probes with world-volume directions as before, $\xi^\alpha = (x^\mu, r, \psi)$, still preserve all of the backgrounds supersymmetries.

Having deformed the original background one turns to the system given by the combined action (1.1). One can anticipate that the brane action will contribute to the energy-momentum tensor in the Einstein equation, add a source term for the 3-form field strength and modify the dilaton equation by a contribution related to the DBI action.

For the case of N_f flavor branes localized at $(\theta_0, \phi_0, \tilde{\theta}_0, \tilde{\phi}_0)$, the brane action is (X^* denoting the pull-back onto the world-volume)

$$S_{\text{Branes}} = T_5 \sum_{N_f} \left(- \int_{\mathcal{M}_6} d^6 \xi e^{\frac{\Phi}{2}} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_6} X^* C_{(6)} \right) \Big|_{(\theta_0, \phi_0, \tilde{\theta}_0, \tilde{\phi}_0)} \quad (2.7)$$

As these branes are localized in the four transverse directions, the equations of motion will contain δ -function sources, making the search for solutions a difficult endeavour. The idea is therefore to smoothly distribute the branes over the transverse directions. If one assumes a transverse brane distribution with density

$$\Omega = \frac{N_f}{(4\pi)^2} \sin \theta \sin \tilde{\theta} d\theta \wedge d\phi \wedge d\tilde{\theta} \wedge d\tilde{\phi} \quad (2.8)$$

the action (2.7) may be generalized to

$$\begin{aligned} S_{\text{Branes}} &= T_5 \left(-\frac{N_f}{(4\pi)^2} \int_{\mathcal{M}_{10}} d^{10}x e^{\frac{\Phi}{2}} \sin \theta \sin \tilde{\theta} \sqrt{-\hat{g}_{(6)}} + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega \right) \\ &= T_5 \left(-\int_{\mathcal{M}_{10}} d^{10}x e^{\frac{\Phi}{2}} \sqrt{-g_{(10)}} |\Omega| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega \right) \end{aligned} \quad (2.9)$$

where we have defined the modulus of a p -form Ω as

$$|\Omega| \equiv \sqrt{\frac{1}{p!} \Omega_{M_1 \dots M_p} \Omega^{M_1 \dots M_p}} \quad (2.10)$$

and have checked the equality of the first and second lines by explicit calculation.

Let us take a look at how the brane action modifies the second order equations of motion, starting with the Ramond-Ramond field strength. Here the relevant part of the total action is

$$S = \int_{\mathcal{M}_{10}} -\frac{1}{2\kappa_{10}^2} \frac{e^{-\Phi}}{2} (F_{(7)} \wedge *F_{(7)}) + T_5 C_{(6)} \wedge \Omega \quad (2.11)$$

If we vary the potential $C_{(6)}$,

$$\begin{aligned} \delta_C S &= \int_{\mathcal{M}_{10}} -\frac{1}{2\kappa_{10}^2} \frac{e^{-\Phi}}{2} (d\delta C_{(6)} \wedge *F_{(7)} + F_{(7)} \wedge *d\delta C_{(6)}) + T_5 \int \delta C_{(6)} \wedge \Omega \\ &= \int_{\mathcal{M}_{10}} \delta C_{(6)} \wedge \left(\frac{1}{2\kappa_{10}^2} d * e^{-\Phi} F_{(7)} + T_5 \Omega \right) \\ \Rightarrow dF_{(3)} &= 2\kappa_{10}^2 T_5 \Omega \end{aligned} \quad (2.12)$$

The change in the dilaton and Einstein equations does not take such a nice geometric form. Choosing $T_5 = \frac{1}{(2\pi)^5}$, $2\kappa_{10}^2 = (2\pi)^7$, the complete equations of motion are

$$\begin{aligned} 0 &= dF_{(3)} - (2\pi)^2 \Omega \\ 0 &= \frac{1}{\sqrt{-g_{(10)}}} \partial_\mu (g^{\mu\nu} \sqrt{-g_{(10)}} \partial_\nu \Phi) - \frac{1}{12} e^\Phi F_{(3)}^2 - \frac{N_f}{8} e^{\frac{\Phi}{2}} \frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}} \sin \theta \sin \tilde{\theta} \\ 0 &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\lambda \Phi \partial^\lambda \Phi \right) \\ &\quad - \frac{1}{12} e^\Phi \left(3F_{\mu\kappa\lambda} F_\nu{}^{\kappa\lambda} - \frac{1}{2} g_{\mu\nu} F_{(3)}^2 \right) - T_{\mu\nu}^{\text{flvr}} \\ T_{\mu\nu}^{\text{flvr}} &= -\frac{N_f}{4} \sin \theta \sin \tilde{\theta} \frac{1}{2} e^{\frac{\Phi}{2}} g_{\mu\alpha} g_{\nu\beta} \hat{g}_{(6)}^{\alpha\beta} \frac{\sqrt{-\hat{g}_{(6)}}}{\sqrt{-g_{(10)}}} \end{aligned} \quad (2.13)$$

The search for solutions of (2.13) is simplified considerably by a powerful result due to Koerber and Tsimpis [38] who showed that any solution to the BPS equations satisfying the modified Bianchi identity of (2.12) solves also the Einstein and dilaton equations and is therefore a solution of (2.13).

So we turn again to the issue of the BPS equations. As the brane embeddings are supersymmetric, the projections (2.5) imposed on the spinor ϵ remain the same. However, the three-form field strength $F_{(3)}$ is modified by the appearance of the source term in (2.13). To incorporate this one makes a new ansatz for the field strength of (2.3)

$$F_{(3)} = -\frac{N_c}{4}e^{-3f-2g-k}e^{123} - \frac{N_f - N_c}{4}e^{-3f-2h-k}e^{\theta\phi 3} \quad (2.14)$$

It follows that the BPS equations (2.6) change to

$$\begin{aligned} 4f &= \Phi \\ h' &= \frac{1}{4}(N_c - N_f)e^{-2h-k} + \frac{1}{4}e^{-2h+k} = \frac{1}{2}e^{3f}F_{\theta\phi 3} + \frac{1}{4}e^{-2h+k} \\ g' &= -N_ce^{-2g-k} + e^{-2g+k} = \frac{1}{2}e^{3f}F_{123} + e^{-2g+k} \\ k' &= \frac{1}{4}(N_c - N_f)e^{-2h-k} - N_ce^{-2g-k} - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\ &= \frac{1}{2}e^{3f}(F_{\theta\phi 3} + F_{123}) - \frac{1}{4}e^{-2h+k} - e^{-2g+k} + 2e^{-k} \\ \Phi' &= -\frac{1}{4}(N_c - N_f)e^{-2h-k} + N_ce^{-2g-k} = -\frac{1}{2}e^{3f}(F_{\theta\phi 3} + F_{123}) \end{aligned} \quad (2.15)$$

It is curious to note that when written in terms of $F_{\theta\phi 3}$ and F_{123} the BPS equations of the deformed and flavored systems are the same – see (2.6) and (2.15). The change in the BPS equations stems solely from the modification of the field strength. This should not come as a surprise, as the brane embeddings are supersymmetric.²

By construction $F_{(3)}$ satisfies the modified Bianchi identity. Thus any solution of (2.15) solves the flavoring problem for the Maldacena-Núñez background. For a discussion of these solutions and their physical interpretation see [16, 17, 18].

In the above background, the generalization of the action (2.7) to (2.9) is fairly intuitive and simple, because there is only one stack of flavor branes with world-volume coordinates that can be globally identified with space-time coordinates. However we can already anticipate the shortcomings of this definition. On a technical level, the first line of (2.9) is inherently dependent on the coordinate split while the second is non-linear in the smearing form Ω . From a more

² Whether the BPS equations are modified by the flavoring procedure is – to some extent – a matter of taste. It depends on whether one makes a sufficiently generic ansatz for the three-form field strength to accommodate the source term. From the perspective of a physicist who is interested in the properties of the dual gauge theory it is more appropriate to consider the BPS equations of the flavored and unflavored theories as different, as some phenomena such as Seiberg duality become apparent at the level of the first-order BPS equations [16]. For a mathematician on the other hand it might be more important to think about the close link between supersymmetry and geometry which is evident in this paper – the fact that the flavor branes are supersymmetric is then reflected by the invariance of the BPS equations in terms of F_{ABC} .

formal point of view it is also unsatisfying that the formalism of those equations treats the DBI and Wess-Zumino contributions to the brane action on an unequal footing. One should recall that, roughly speaking, the DBI action defines the tree level couplings of the brane to the NS sector of the background while the couplings to Ramond-Ramond fields are contained in the Wess-Zumino term. A standard string theory calculation shows the cancellation of the effects of closed strings from the two sectors on supersymmetric branes. So it would be desirable to see an explicit symmetry between the two terms even after smearing. Adopting once again a more physics centered perspective we might also wonder if there are any constraints on the choice of the smearing form. E.g. one should note that the smearing form does not agree with the volume form induced on the four-cycle $(\theta, \phi, \tilde{\theta}, \tilde{\phi})$. At first glance it might appear that there are none. After all, the cancellations between parallel BPS branes allow us to place them at arbitrary separations. As we will soon see, however, there are constraints on Ω which can be traced back to the geometric structure of the background.

The perspective of generalized calibrated geometry The properties of generalized calibrations and their relation to supersymmetry are discussed in detail in appendix A. As the backgrounds considered in this paper are not fully generic, yet only include dilaton and Ramond-Ramond fields in type IIB supergravity, we will not make use of the most general concept of a generalized calibration. Again we refer to [32, 33]. For our purposes it is sufficient to think of calibrations as $(p+1)$ -forms $\hat{\phi}$, such that a p -brane with embedding $X^M(\xi)$ is supersymmetric if and only if it satisfies

$$X^* \hat{\phi}_{(p+1)} = \sqrt{-\hat{g}_{(p+1)}} d^{p+1} \xi \quad (2.16)$$

As discussed in the appendix, this can be understood as a simple rephrasing of the κ -symmetry condition on the SUSY spinor ϵ ,

$$\Gamma_\kappa \epsilon = \epsilon \quad (2.17)$$

For us the most interesting feature of (2.16) is that when pulled back onto the world-volume of the brane, the calibration form is equivalent to the induced volume form, and one may write the DBI action as

$$S_{DBI} = -T_p \int_{\mathcal{M}_{p+1}} e^{\frac{p-3}{4}\Phi} X^* \hat{\phi} \quad (2.18)$$

Furthermore, if the p -brane couples electrically to the flux given by $F_{(p+2)}$, supersymmetry in the Einstein frame requires [31]

$$d(e^{\frac{p-3}{4}\Phi} \hat{\phi}) = F_{(p+2)} \quad (2.19)$$

In the case at hand, the calibration six-form is given by

$$\hat{\phi} = \frac{1}{6!} (\epsilon^\dagger \sigma_3 \otimes \Gamma_{a_0 \dots a_5} \epsilon) e^{a_0 \dots a_5} \quad (2.20)$$

As explained in appendix B, evaluation of the calibration form requires only the chirality of the type IIB spinors, $\epsilon = \Gamma^{11} \epsilon$ and knowledge of the projections

imposed on the SUSY spinors (2.5). From the last of these it follows that one of the Majorana-Weyl spinors of type IIB is fixed to zero, $\epsilon = \begin{pmatrix} \epsilon \\ 0 \end{pmatrix}$. Thus there is only one calibration six-form and we may use ϵ instead of ϵ . In section 3 we will encounter an example with two calibration forms. Combining the SUSY projections (2.5) with the definition (2.20) yields

$$\hat{\phi}_{x^0 x^1 x^2 x^3 \theta \phi} = \epsilon^\dagger \Gamma_{x^0 x^1 x^2 x^3 \theta \phi} \epsilon = -\epsilon^\dagger \Gamma_{r123} \epsilon = -1 \quad (2.21)$$

The second equality makes use of chirality, the third of the SUSY projections and the normalization $\epsilon^\dagger \epsilon = 1$. When calculating calibration forms it is actually more difficult to show that certain components vanish. However, the process is rather straightforward and discussed in considerable detail in appendix B. When the dust settles, we are left with

$$\hat{\phi} = e^{x^0 x^1 x^2 x^3} \wedge (e^{r3} - e^{\theta \phi} - e^{12}) \quad (2.22)$$

As e^3 is the only part of the vielbein containing $d\psi$, it is obvious that equation (2.16) is satisfied and we recover the result of [37] that the embedding in question is supersymmetric. Noting that

$$\begin{aligned} \sqrt{-\hat{g}_{(6)}} d^6 \xi &= e^{x^0 x^1 x^2 x^3 r3} \\ \Omega &= 4N_f e^{-4f-2g-2h} e^{\theta \phi 12} \end{aligned} \quad (2.23)$$

it is easy to see that we may write the smeared brane action (2.9) as

$$S_{\text{branes}} = T_5 \int_{\mathcal{M}_{10}} (-e^{\frac{\Phi}{2}} \hat{\phi} + C_{(6)}) \wedge \Omega \quad (2.24)$$

In opposite to (2.9) this is independent of coordinates, linear in the smearing form, and treats the DBI and Wess-Zumino contributions to the brane action on an equal footing.

Concerning the supersymmetry condition (2.19), we find

$$\begin{aligned} d(e^{\frac{\Phi}{2}} \hat{\phi}) &= e^{-f+\frac{\Phi}{2}} e^{x^0 x^1 x^2 x^3} \wedge [e^{-2g}(2e^k - 6e^{2g} f' - 2e^{2g} g' - e^{2g} \Phi') e^{r12} \\ &\quad + e^{-2h} (\frac{1}{2} e^k - 6e^{2h} f' - 4e^{2h} h' - e^{2h} \Phi') e^{r\theta \phi}] \end{aligned} \quad (2.25)$$

Using the BPS equations (2.6) or (2.15), one may verify for the three-form field strength with (2.14) and without sources (2.3) that $d(e^{\frac{\Phi}{2}} \hat{\phi}) = F_{(7)}$ is satisfied. We can exploit the calibration form even further. From $e^{-\Phi} * F_{(7)} = F_{(3)}$ and $dF_{(3)} = (2\pi)^2 \Omega$ it follows that

$$\begin{aligned} e^{-\Phi} * d(e^{\frac{\Phi}{2}} \hat{\phi}) &= F_{(3)} \\ d[e^{-\Phi} * d(e^{\frac{\Phi}{2}} \hat{\phi})] &= (2\pi)^2 \Omega \end{aligned} \quad (2.26)$$

Again note that these equations hold with or without the backreaction of the source terms – in the latter case with $\Omega = 0$. One should think of them rather as a characteristic of the supersymmetries preserved by the background than a property of the branes.

When we first introduced the smearing form in (2.8) it appeared that its choice was rather arbitrary. After all supersymmetry allows us to place branes

at arbitrary separations. However, (2.26) is not a result of supersymmetry alone yet rather an interplay of supersymmetry and the Einstein equations, as the following illustrates.

$$d(e^{\frac{\Phi}{2}}\hat{\phi}) \stackrel{\text{SUSY}}{=} F_{(7)}, \quad *e^{-\Phi}F_{(7)} \equiv F_{(3)}, \quad dF_{(3)} \stackrel{\text{EOM}}{=} (2\pi)^2\Omega \quad (2.27)$$

BPS equations and G -structures We showed before that the requirement of supersymmetry is related to geometry, notably with the calibration form. As supersymmetry gives us the BPS equations of the system, it is logical to think that one can retrieve those equations through geometric considerations, namely G -structures. When looking at the supersymmetric gravitino equation, we can identify $F_{(3)}$ with a torsion (straightforward in string frame), defining a new covariant derivative $\tilde{\nabla}_\mu$ such that

$$\tilde{\nabla}_\mu \epsilon = \epsilon \quad (2.28)$$

This means that we have a covariantly constant spinor satisfying certain projections (2.5). $\epsilon = \sigma_3 \epsilon$ states that there is only one structure. The other two tell us that in the six-dimensional internal manifold, there is a covariantly constant complex chiral spinor η verifying

$$\gamma_{r123}\eta = \eta \quad \gamma_{r\theta\phi 3}\eta = \eta \quad (2.29)$$

where γ_i are the gamma matrices of the six-dimensional internal manifold. We can choose the chirality of η to be

$$\nu\gamma_{r123\theta\phi}\eta = -\eta \quad (2.30)$$

Then we recognize that the six-dimensional manifold is a generalized Calabi-Yau. It has a Kähler two-form J and a holomorphic three-form Ω defined as

$$J_{mn} = \eta^\dagger \gamma_{mn} \eta \quad (2.31)$$

$$\Omega_{mnp} = \eta^T \gamma_{mnp} \eta \quad (2.32)$$

Supersymmetry imposes the following conditions on the forms (see [35]):

$$d(e^\Phi *_6 J) = 0 \quad (2.33)$$

$$d(e^{\frac{5}{4}\Phi} \Omega) = 0 \quad (2.34)$$

From those equations, plus the generalized calibration condition (2.26), we can retrieve the BPS equations of the system, imposing $4f = \Phi$. Indeed, this last condition, describing how the internal manifold is embedded in space-time, cannot be captured by those geometric properties that concern only the six-dimensional manifold. It can however easily be found using the supersymmetric variations of the dilatino and the gravitino.

2.1.2 An $\mathcal{N} = 1$, $d = 2 + 1$ example

We turn now to the string dual of a $d = 2 + 1$ dimensional $\mathcal{N} = 1$ theory that was discussed in [25]. We will leave the discussion rather brief, only exhibiting the equivalence of the actions (2.9) and (2.24) for this example. In comparison

to the $\mathcal{N} = 1$ sQCD-like dual of the previous section the situation is complicated by the fact that there are three stacks of branes. While it is possible to find coordinates such that the worldvolume of one of these stacks may be identified with space-time coordinates, it is not possible to do so for all three stacks simultaneously. The system has the topology $\mathbb{R}^{1,2} \times \mathbb{R} \times S^3 \times S^3$. As in section 2.1.1, we shall work with a simplification, the truncated system, for which the background is given by

$$\begin{aligned} e^{x^i} &= e^f dx^i & e^r &= e^f dr & e^i &= \frac{e^{f+h}}{2} \sigma^i & e^{\hat{i}} &= \frac{e^{f+g}}{2} (\omega^i - \frac{1}{2} \sigma^i) \\ F_{(3)} &= -2N_c e^{-3g-3f} e^{\hat{1}\hat{2}\hat{3}} + \frac{1}{2} N_c e^{-g-2h-3f} (e^{13\hat{2}} - e^{12\hat{3}} - e^{23\hat{1}}) \end{aligned} \quad (2.35)$$

σ^i and ω^i are sets of Maurer-Cartan forms parametrizing the two three-spheres. The projections satisfied by the SUSY spinor η are

$$\Gamma_{1\hat{1}2\hat{2}}\eta = -\eta \quad \Gamma_{1\hat{1}3\hat{3}}\eta = -\eta \quad \Gamma_{2\hat{2}3\hat{3}}\eta = -\eta \quad \Gamma_{r\hat{1}\hat{2}\hat{3}}\eta = \eta \quad \eta = \sigma_3\eta \quad (2.36)$$

And the BPS equations take the form

$$\begin{aligned} \Phi' &= N_c e^{-3g} - \frac{3}{4} N_c e^{-g-2h} \\ h' &= \frac{1}{2} e^{g-2h} + \frac{1}{2} N_c e^{-g-2h} \\ g' &= e^{-g} - \frac{1}{4} e^{g-2h} + \frac{N_c}{4} e^{-g-2h} - N_c e^{-3g} \\ \Phi &= 4f \end{aligned} \quad (2.37)$$

Once more, it follows from $\eta = \sigma_3\eta = \begin{pmatrix} \eta \\ 0 \end{pmatrix}$ that there is only one calibration six-form which is given by (assuming $\Gamma^{11}\eta = -\eta$)

$$\hat{\phi} = e^{012} \wedge (e^{r1\hat{1}} + e^{r2\hat{2}} + e^{r3\hat{3}} - e^{123} + e^{3\hat{1}\hat{2}} - e^{2\hat{1}\hat{3}} + e^{1\hat{2}\hat{3}}) \quad (2.38)$$

From the calibration condition for supersymmetric branes, $X^*\hat{\phi} = d\xi^6 \sqrt{-\hat{g}_{(6)}}$, one can see immediately that there are supersymmetric 5-brane embeddings with tangent vectors³ $(\partial_{x^0}, \partial_{x^1}, \partial_{x^2}, E_r, E_i, E_{\hat{i}})$, $i \in \{1, 2, 3\}$. We also learn from (2.38) that these embeddings are absolutely equivalent. They were originally derived in [25] using κ -symmetry. There the authors introduced a standard set of Maurer-Cartan forms ω, σ to parametrize the two S^3 s, and then found a coordinate representation of the $(\partial_\mu, \partial_r, E_3, E_{\hat{3}})$ branes given by $(x^\mu, r, \psi_1, \psi_2)$.

³ When labeling brane embeddings in terms of their tangent vectors one should think of the brane being along the submanifold spanned by the integral curves of the tangent vector fields. That is, if one were to find coordinates y^M such that

$$\partial_{x^0} = \partial_{y^0} \quad \partial_{x^1} = \partial_{y^1} \quad \partial_{x^2} = \partial_{y^2} \quad E_r = \partial_{y^3} \quad E_1 = \partial_{y^4} \quad E_{\hat{1}} = \partial_{y^5}$$

the corresponding $1\hat{1}$ brane embedding would be given by

$$Y^\alpha(\xi) = \xi^\alpha \quad Y^a = \text{const.} \quad \alpha \in \{0, \dots, 5\} \quad a \in \{6, \dots, 9\}$$

One should note however, that it is necessary to verify, that the distribution given by the tangent vectors is integrable, i.e. to verify that the coordinates y^M exist. One can do so using Frobenius theorem, which states that a distribution given by vectors T_a is integrable iff it is in involution, that is iff $[T_a, T_b] = f_{abc} T_c$.

Subsequently they argued from the symmetries of the space that there are also $1\hat{1}$ and $2\hat{2}$ embeddings, whose coordinate representation would become apparent upon using different Maurer-Cartan forms. As we mentioned earlier, it does not seem to be possible to find global coordinates for this system in which all three flavor brane embeddings have good coordinate representations – thus this is an ideal setting for using the calibration form (2.38).

Our analysis here shall start with the $3\hat{3}$ embeddings. In [25] their smeared action was given by

$$\begin{aligned}
S_{D5} &= T_5 \left(- \int d^{10}x e^{\frac{\Phi}{2}} \sqrt{-G_{10}} |\Omega^{(1)}| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega^{(1)} \right) \\
\Omega^{(1)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{12\hat{1}\hat{2}} \\
|\Omega^{(1)}| &= \frac{N_f}{\pi^2} e^{-4f-2h-2g} \\
\sqrt{-G_{10}} &= \frac{1}{64} e^{10f+3g+3h} \sin \theta \sin \tilde{\theta} \\
\sqrt{-\hat{G}_6} &= \frac{1}{4} e^{6f+g+h}
\end{aligned} \tag{2.39}$$

Now

$$\hat{\phi} \wedge \Omega^{(1)} = -\frac{N_f}{\pi^2} e^{-4f-2h-2g} \sqrt{-G_{10}} d^{10}x = d^{10}x \sqrt{-G_{10}} |\Omega^{(1)}| \tag{2.40}$$

Thus again, we may write the action of one stack of $(3\hat{3})$ branes as

$$S_{D5} = T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}} \hat{\phi} + C_{(6)} \right) \wedge \Omega^{(1)} \tag{2.41}$$

The above may be easily generalized to the case of three stacks of D5-branes as the expression is linear in Ω .

$$\begin{aligned}
S_{D5} &= T_5 \int_{\mathcal{M}_{10}} \left(-e^{\frac{\Phi}{2}} \hat{\phi} + C_{(6)} \right) \wedge \Omega \\
\Omega &= \Omega^{(1)} + \Omega^{(2)} + \Omega^{(3)} \\
\Omega^{(2)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{13\hat{1}\hat{3}} \\
\Omega^{(3)} &= -\frac{N_f}{\pi^2} e^{-4f-2h-2g} e^{23\hat{2}\hat{3}}
\end{aligned} \tag{2.42}$$

Where $\Omega^{(2)}$ is the smearing form for branes extending along $2\hat{2}$ and $\Omega^{(3)}$ smears the $1\hat{1}$ embedding. The linearity of the above expression gives a good motivation for the use of $\sum_i |\Omega^{(i)}|$ instead of $|\Omega|$ in the original action of [25]

$$S_{D5} = T_5 \left(- \int d^{10}x e^{\frac{\Phi}{2}} \sqrt{-G_{10}} \sum_{i=1}^3 |\Omega^{(i)}| + \int_{\mathcal{M}_{10}} C_{(6)} \wedge \Omega \right) \tag{2.43}$$

Independently of whether one uses the action (2.42) or (2.43) the Bianchi identity is modified to $dF_{(3)} = -2\kappa_{10}^2 T_5 \Omega$ – the minus sign being due to the

convention $e^\Phi F_{(3)} = - * F_{(7)}$ used in [25]. Accordingly one changes the ansatz for the field-strength by adding a term $f_{(3)}$ which is not closed,

$$\begin{aligned} F_{(3)} &\mapsto F_{(3)} + f_{(3)} \\ f_{(3)} &= 2N_f e^{-g-2h-3f} (e^{12\hat{3}} + e^{23\hat{1}} - e^{13\hat{2}}) \end{aligned} \quad (2.44)$$

The BPS equations (2.37) change to

$$\begin{aligned} \Phi' &= N_c e^{-3g} - \frac{3}{4}(N_c - N_f) e^{-g-2h} \\ h' &= \frac{e^{g-2h}}{2} + \frac{N_c - 4N_f}{2} e^{-g-2h} \\ g' &= e^{-g} - \frac{1}{4} e^{g-2h} - N_c e^{-3g} + \frac{N_c - 4N_f}{4} e^{-g-2h} \\ \Phi &= 4f \end{aligned} \quad (2.45)$$

Let us now turn to the SUSY condition (2.19). A straightforward yet tedious calculation yields

$$\begin{aligned} d(e^{\frac{\Phi}{2}} \hat{\phi}) &= e^{\frac{\Phi}{2}-f} e^{012} \wedge \{ (2e^{-g} - 6f' - 2g' - h' - \Phi') (e^{r1\hat{2}\hat{3}} - e^{r2\hat{1}\hat{3}} + e^{r3\hat{1}\hat{2}}) \\ &\quad + \frac{e^{-2h}}{2} (-3e^g + 12e^{2h} f' + 6e^{2h} h' + e^{2h} \Phi') e^{r123} \} \end{aligned} \quad (2.46)$$

Using the BPS equations (2.37) or (2.45) respectively one can verify that $-e^{-\Phi} * d(e^{\frac{\Phi}{2}} \hat{\phi}) = F_{(3)}$ is satisfied in both the deformed and flavored case. Furthermore we know that $dF_{(3)} = (2\pi)^2 \Omega$, thus we are again able to obtain a constraint on the smearing form as

$$(2\pi)^2 \Omega = d[-e^{-\Phi} * d(e^{\frac{\Phi}{2}} \hat{\phi})] \quad (2.47)$$

We immediately see why there have to be three stacks of flavor D5-branes in the backreacted solution – the calibration form respects the symmetries of the two three-spheres and from (2.47) it follows that the same holds true for the smearing form. It would therefore not be possible to obtain a smeared system with only one or two of the three stacks.

We can again use G -structures to derive the BPS equations for the system. In this case the internal manifold is seven-dimensional, with a covariantly constant spinor which satisfies

$$\gamma_{1\hat{1}2\hat{2}} \eta = -\eta \quad \gamma_{1\hat{1}3\hat{3}} \eta = -\eta \quad \gamma_{r\hat{1}2\hat{3}} \eta = \eta \quad (2.48)$$

We recognize here a generalized G_2 holonomy manifold with the associative three-form $\hat{\phi}$ defined as

$$\hat{\phi}_{mnp} = -i\bar{\eta} \gamma_{mnp} \eta \quad (2.49)$$

The condition imposed by supersymmetry is

$$d(e^\Phi *_7 \hat{\phi}) = 0 \quad (2.50)$$

Together with the generalized calibration condition, and assuming $\Phi = 4f$, this condition provides us with a method to rederive the BPS equations (2.37), (2.45).

2.1.3 The Klebanov-Witten model

Finally we take a look at the Klebanov-Witten model for the cases of massless [19] and massive flavors [24]. The Klebanov-Witten model [3] is based on D3-branes at the tip of the conifold and is dual to a certain $\mathcal{N} = 1$ super Yang-Mills theory. So apart from the dilaton and the metric there is self-dual $F_{(5)}$ flux due to the D3s. In contrast to the previous two examples, one uses D7s to introduce flavor degrees of freedom into the system. These source $F_{(1)}$, so the suitable ansatz for the relevant deformed, flavored background is

$$\begin{aligned}
ds^2 &= h^{-\frac{1}{2}} dx_{1,3}^2 \\
&+ h^{\frac{1}{2}} \left[e^{2f} d\rho^2 + \frac{e^{2g}}{6} \sum_{i=1,2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{e^{2f}}{9} (d\psi + \sum_{i=1,2} \cos \theta_i d\phi_i)^2 \right] \\
F_{(5)} &= 27\pi N_c e^{-4g-f} h^{-5/4} (e^{x^0 x^1 x^2 x^3 \rho} - e^{\theta_1 \phi_1 \theta_2 \phi_2 \psi}) \\
F_{(1)} &= \frac{N_f(\rho)}{4\pi} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)
\end{aligned} \tag{2.51}$$

with $\psi \in [0, 4\pi]$, $\theta_i \in [0, \pi]$, $\phi_i \in [0, 2\pi]$, $\rho \in \mathbb{R}$. There is an obvious choice of vielbein

$$\begin{aligned}
e^{x^i} &= h^{-1/4} dx^i & e^\rho &= h^{1/4} e^f d\rho \\
e^{\theta_i} &= \frac{1}{\sqrt{6}} h^{1/4} e^g d\theta_i & e^{\phi_i} &= \frac{1}{\sqrt{6}} h^{1/4} e^g \sin \theta_i d\phi_i \\
e^\psi &= \frac{1}{3} h^{1/4} e^f (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)
\end{aligned} \tag{2.52}$$

The flavor branes behave differently in the massless or massive case. In the former, the authors of [19] used two stacks of branes whose world-volume coordinates may once more be identified with space-time ones,

$$\begin{aligned}
\xi_1^\alpha &= (x^\mu, \rho, \theta_2, \phi_2, \psi) & \theta_1 &= \text{const.} & \phi_1 &= \text{const.} \\
\xi_2^\alpha &= (x^\mu, \rho, \theta_1, \phi_1, \psi) & \theta_2 &= \text{const.} & \phi_2 &= \text{const.}
\end{aligned} \tag{2.53}$$

So prior to smearing the system has a global $\mathcal{U}(N_f) \times \mathcal{U}(N_f)$ flavor symmetry – one for each set of D7s. This is obviously a four-parameter family of embeddings, which can be smeared over the transverse (θ_i, ϕ_i) directions. In the massive case the embeddings are more complicated. In the field theory, the mass term breaks the global symmetry to the diagonal $\mathcal{U}(N_f) \times \mathcal{U}(N_f) \mapsto \mathcal{U}(N_f)$, which corresponds the two stacks joining into one on the string theory side. There is again a four-parameter family of brane embeddings, yet as the generic embedding is much more complicated than those of (2.53), we shall only look at one representative, trusting that the calibration form will ensure that we make use of the whole family of branes. Choosing world-volume coordinates $\xi = (x^\mu, \theta_1, \phi_1, \theta_2, \phi_2)$, this is given by

$$\begin{aligned}
X^M(\xi) &= \left(x^\mu, \rho_q - \frac{2}{3} \log \sin \frac{\theta_1}{2} - \frac{2}{3} \log \sin \frac{\theta_2}{2}, \theta_1, \phi_1, \theta_2, \phi_2, \phi_1 + \phi_2 + 2\beta \right) \\
\rho_q, \beta &= \text{const.}
\end{aligned} \tag{2.54}$$

The constant ρ_q denotes the minimal radius reached by the brane and may therefore be identified as the mass.

The branes have an $(7+1)$ -dimensional world-volume and we therefore need to construct the calibration 8-form. In the case at hand this requires the knowledge of the supersymmetric spinors on the conifold. These were discussed in [39]. Our conventions however are those of [19]. The SUSY spinor ϵ is related to a constant spinor η as $\epsilon = h^{-1/8} e^{-\frac{1}{2}\psi} \eta$. Both satisfy the projections

$$\begin{aligned} \iota\sigma_2 \otimes \Gamma_{x^0 x^1 x^2 x^3} \eta &= \eta & \Gamma_{r\psi} &= \iota\sigma_2 \eta \\ \Gamma_{\theta_1 \phi_1} &= -\iota\sigma_2 \eta & \Gamma_{\theta_2 \phi_2} &= -\iota\sigma_2 \eta \end{aligned} \quad (2.55)$$

From equation (A.5) it follows that the calibration form for D7-branes is given by

$$\hat{\phi} = \frac{1}{8!} (\eta^\dagger \iota\sigma_2 \otimes \Gamma_{a_0 \dots a_7} \eta) e^{a_0 \dots a_7} \quad (2.56)$$

which we may evaluate using (2.55) to be

$$\hat{\phi} = e^{x^0 x^1 x^2 x^3} \wedge (e^{\rho\theta_1 \phi_1 \psi} + e^{\rho\theta_2 \phi_2 \psi} - e^{\theta_1 \phi_1 \theta_2 \phi_2}) \quad (2.57)$$

At this point we may calculate the pull-backs $X^* \hat{\phi}$ for both embeddings (2.53) and (2.54). Finding $X^* \hat{\phi} = \sqrt{-\hat{g}_{(8)}} d^8 \xi$ we do thus verify that the brane embeddings are indeed supersymmetric.

In Einstein frame, the integrand of the DBI action is $e^\Phi \sqrt{-\hat{g}_{(8)}} d^8 \xi = e^\Phi X^* \hat{\phi}$. As before, supersymmetry requires this to satisfy $d(e^\Phi \hat{\phi}) = F_{(9)}$. Making use of the definition $F_{(1)} = -e^{-2\Phi} * F_{(9)}$ and the equation of motion $dF_{(1)} = -\Omega$, we arrive at the following

$$\begin{aligned} F_{(9)} &= d(e^\Phi \hat{\phi}) = 3h^{-\frac{1}{4}} e^{-f} \frac{N_f(\rho)}{4\pi} e^{x^0 x^1 x^2 x^3 \rho \theta_1 \phi_1 \theta_2 \phi_2} \\ F_{(1)} &= -e^{-2\Phi} * F_{(9)} = -3h^{-\frac{1}{4}} e^{-f} \frac{N_f(\rho)}{4\pi} e^\psi \\ \Omega &= -dF_{(1)} = \frac{N_f(\rho)}{4\pi} (\sin \theta_1 d\theta_1 \wedge d\phi_1 + \sin \theta_2 d\theta_2 \wedge d\phi_2) \\ &\quad + \frac{N'_f(\rho)}{4\pi} d\rho \wedge (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \\ N_f(\rho) &= \frac{4\pi}{3} e^{-2g-\Phi} (4e^{2g} g' + e^{2g} \Phi' - 4e^{2f}) \end{aligned} \quad (2.58)$$

The name for the function $N_f(\rho)$ has been chosen in anticipation of what is to come – it will denote the effective number of flavors at a given energy scale. It should not be confused with N_f , the number of flavor branes.

One should notice that the only assumptions made in deriving (2.58) are the form of $F_{(5)}$ and the vielbein describing the deformed background (2.52). That is, the above relations hold for all types of D7-branes one might want to smear, massless or massive. They allow us to write down the BPS equations of the system which can be derived from the SUSY variations [24] or using geometric methods.

$$\begin{aligned} g' &= e^{2f-2g} & f' &= 3 - 2e^{2f-2g} - \frac{3N_f(\rho)}{8\pi} e^\Phi \\ \Phi' &= \frac{3N_f(\rho)}{4\pi} e^\Phi & h' &= -27\pi N_c e^{-4g} \end{aligned} \quad (2.59)$$

Note that there are four first-order equations for the five functions Φ, f, g, h, N_f . Furthermore, the smearing procedure always uses the same action,

$$S_{\text{Branes}} = T_7 \int_{\mathcal{M}_{10}} \left(-e^{\Phi} \hat{\phi} + C_{(8)} \right) \wedge \Omega \quad (2.60)$$

The authors of [19, 24] used an action of the type encountered in (2.9) and (2.43), yet once more the equivalence with (2.60) may be shown explicitly – we will also present a general proof of the validity of (2.60) in section 2.2.

Given that the discussion up to this point is completely independent of the type of brane one wants to smear, one might ask how to distinguish between the different classes of potential flavor branes. The answer to that question lies in the choice of the function $N_f(\rho)$.

However, even before looking at specific choices of $N_f(\rho)$ the generic form of Ω in (2.58) tells us quite a bit about possible smeared-brane configurations. For once, it is not possible to break the $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry of the background, as this is the inherent symmetry of Ω (The \mathbb{Z}_2 describes the exchange of the two spheres). So for massless branes we will only be able to smear both stacks simultaneously.

The massless branes may be identified with the coordinates given by (2.53). Thus they are smeared by the terms proportional to $d\theta_i \wedge d\phi_i$. As the smearing form is symmetric under the exchange $(\theta_1, \phi_1) \leftrightarrow (\theta_2, \phi_2)$ it is clear that we will have to smear both stacks of branes. I.e. one cannot assume $\Omega_{\theta_1 \phi_1}$ to vanish without $\Omega_{\theta_2 \phi_2}$ vanishing as well. The term involving $d\rho$ on the other hand is not transverse to the world-volume defined by (2.53). In order to smear only massless branes, one needs this term to vanish. I.e. massless branes require

$$N'_f(\rho) = 0 \quad (2.61)$$

Using this constraint the system (2.59) is fully determined and can be solved. In that case, we can see from (2.60) that the last term in (2.57) – which does not contain e^ρ – does not contribute. Interpreting the smearing form as a brane-density, we may identify the overall factor with the number of flavors,

$$N_f = 4\pi N_f(\rho) \quad (2.62)$$

That is, our decision to smear N_f massless branes with a constant number of flavors imposes two constraints into the system, namely (2.61) and (2.62).

Our choice for $N_f(\rho)$ may also be interpreted using the local geometry of the brane embeddings instead of their global coordinates. The vectors

$$(\partial_{x^\mu}, \partial_\rho, \partial_\psi) \quad (2.63)$$

are tangent to either stack of branes. As the smearing form should – locally – define a volume orthogonal to these vectors, we demand⁴

$$\iota_{\partial_{x^\mu}} \Omega = \iota_{\partial_\rho} \Omega = \iota_{\partial_\psi} \Omega = 0 \quad (2.64)$$

⁴ Interior multiplication of forms with vectors is defined as

$$(\iota_X \omega)_{N_1 \dots N_{p-1}} = X^M \omega_{MN_1 \dots N_{p-1}}$$

It follows that $4\pi N_f(\rho) = \text{const.} = N_f$.

Turning to the massive case, the authors of [24] used

$$N'_f(\rho) = 3N_f e^{3\rho_q - 3\rho} (3\rho - 3\rho_q) \quad (2.65)$$

In principle one would expect that one can combine the knowledge of the embedding (2.54) together with the general form for Ω in order to derive this form for $N_f(\rho)$, as we did for massless branes, yet we were unable to do so. The reason might be that the authors of [24] considered not just the single representative of the family of massive embeddings, yet a complete distribution. Our analysis contributes to the construction of $N_f(\rho)$ in so far, however, as the derivation in [24] requires the assumption that the $SU(2) \times SU(2) \times U(1) \times \mathbb{Z}_2$ symmetry cannot be broken, while we have shown that this is not an assumption, but an innate property of the background.

Once more one invokes [38] and needs only to study the BPS equations (2.59) together with the modified Bianchi identity to find solutions of the second order equations. We refer to the original papers for a discussion of the solutions.

Anticipating the possibility of using the formalism presented up to this point in order to smear branes whose coordinate representation is unknown, we shall now discuss the problem of correctly interpreting the smearing form Ω . Using the vielbein it takes the form

$$\Omega = \frac{6N_f(\rho)}{\sqrt{h}} e^{-2g}(e^{\theta_1\phi_1} + e^{\theta_2\phi_2}) + \frac{6N'_f(\rho)}{\sqrt{h}} e^{-2f} e^{\rho\psi} \quad (2.66)$$

In the case for the massless embeddings (2.53) the second term disappeared and it is straightforward to interpret the first as a distribution on the space transverse to the two stacks of D7s. If we did not know about the massive embeddings (2.54) it would be tempting to interpret the term including N'_f as the distribution of a third stack of branes extending along x^μ , wrapping $(\theta_1, \phi_1, \theta_2, \phi_2)$ and positioned at fixed (ρ, ψ) . That is we would think of this term as a contribution of compact, smeared D7 branes. The presence of such branes is potentially disastrous as the gauge theory in their world-volume could remain dynamic from a four-dimensional point of view. In the case at hand, the eight-dimensional gauge coupling behaves as $g_{\text{YM}} \sim g_s \alpha'^2$, which vanishes for $\alpha' \rightarrow 0$, the decoupling limit of the D3s. When using D5 branes on the other hand this does not have to happen. For the massive Klebanov-Witten model we know that our interpretation in terms of compact D7 branes is wrong as we are smearing a single stack of massive ones. Keeping this in mind we conclude that it is not straightforward to know which branes have been smeared by simply investigating Ω .

Again we would like to carry on the procedure we used previously to find the BPS equations (2.59) through geometric properties. However, in the previous examples, the starting point was to identify $F_{(3)}$ with a torsion. In the Klebanov-Witten model, there is no $F_{(3)}$ but instead $F_{(1)}$ and $F_{(5)}$. As a consequence, it is not straightforward to transform the supersymmetric gravitino variation into a covariant derivative. In this case as in the other ones, supersymmetry should nevertheless impose conditions on the geometry of the internal manifold. Understanding how to derive those conditions is left for future work.

2.2 The generic case

The three examples of the previous section provide us with all the intuition needed to understand the relation between generalized calibrated geometry and supergravity duals with backreacted, smeared flavors. For a type IIA/B background with Ramond-Ramond flux $F_{(p+2)}$ and arbitrary dilaton we expect that we should always be able to write the action in terms of the calibration and smearing form as

$$S_{\text{Branes}} = -T_p \int_{\mathcal{M}_{10}} (e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)}) \wedge \Omega \quad (2.67)$$

Now as we discussed in appendix A, supersymmetry imposes

$$d(e^{\frac{p-3}{4}\Phi} \hat{\phi}) = F_{(p+2)} \quad (2.68)$$

Combining this with the modified p -form equation of motion $dF_{(10-p-2)} = 2\kappa_{10}^2 T_p \Omega$, as derived in (2.12), we may link the calibration and the smearing form

$$d[*e^{\frac{10-2p-4}{4}\Phi} d(e^{\frac{p-3}{4}\Phi} \hat{\phi})] = \pm 2\kappa_{10}^2 T_p \Omega \quad (2.69)$$

The overall sign depends on the conventions used when relating the field strength $F_{(p+2)}$ to its dual. In what follows, we shall give a more formal argument why the action (2.67) is appropriate to describe smeared branes, show that it is equivalent to the actions previously used in the literature and finally examine some of the consequences of the above relations.

2.2.1 The smeared brane action

The problem of smearing a generic DBI+Wess-Zumino system takes a rather simple form from a mathematical point of view. Here we are dealing with two spaces, the world-volume \mathcal{M}_{p+1} and space-time \mathcal{M}_{10} , which are related by the embedding map

$$\begin{aligned} X : \mathcal{M}_{p+1} &\rightarrow \mathcal{M}_{10} \\ \xi^\alpha &\mapsto X^M(\xi) \end{aligned} \quad (2.70)$$

As integrals of scalars are ill-defined on manifolds, it is mandatory for this discussion to think of the brane action as an integral of differential forms. For the Wess-Zumino term, the integrand is the pull-back of the relevant electrically coupled gauge-potential onto the world-volume, $\int_{\mathcal{M}_{p+1}} X^* C_{(p+1)}$. Whereas we integrate over the induced volume form and the dilaton in the case of the DBI action,⁵ $\int_{\mathcal{M}_{p+1}} d^{p+1} \xi e^{\frac{p-3}{4}\Phi} \sqrt{-\hat{g}_{(p+1)}}$. The crucial point is that there is no way to a priori identify the DBI integrand with a $(p+1)$ -form in space-time, as the induced volume form is usually not thought of as the pull-back of a differential form. Indeed, we were rather careless in section 2.1 as we did not discriminate between the set of form-fields in the world-volume of the brane, $\Omega(\mathcal{M}_{p+1})$, and that defined on all of space-time, $\Omega(\mathcal{M}_{10})$.

One might argue that we should be able to somehow push the induced volume form forward onto space-time. This is certainly the case if we are able to identify

⁵The discussion in this section considers branes without world-volume gauge fields or the NS potential B . See however [23, 32, 33].

world-volume with space-time coordinates. In the case of the string dual of the $\mathcal{N} = 1$ sQCD-like theory this was strikingly obvious. As a matter of fact, the action written in the first line of (2.9) is exactly of the form (2.67). In a generic situation however, we cannot expect to be able to find such a set of global coordinates. Moreover the natural operations induced by maps between manifolds are push-forwards of vectors and pull-backs of forms. And as they connect spaces of different dimensions, they cannot be assumed to be invertible.

This is where calibrated geometry comes in. As we have seen before, supersymmetric branes satisfy $X^*\hat{\phi} = \sqrt{-\hat{g}_{(p+1)}}d^{p+1}\xi$. Making use of this fact allows us to treat the DBI and Wess-Zumino terms on a democratic footing, as both integrands can now be written as pull-backs of $(p+1)$ -forms defined on space-time.

We shall now show that the action (2.67) can always be written in the form used in [19, 25]. Essentially the whole discussion boils down to the fact that we may locally choose nice coordinates. Let us assume that we have a single stack of supersymmetric p -branes. Locally, we may choose coordinates $x^M = (z^\mu, y^m)$ such that the branes extend along the z^μ ; that is for world-sheet coordinates ξ^μ and embeddings $X^M(\xi)$ we have

$$\partial_\nu X^M = \begin{cases} \delta_\nu^M & M \in \{0, \dots, p\} \\ 0 & M \notin \{0, \dots, p\} \end{cases} \quad (2.71)$$

The vectors ∂_μ are tangent to the brane. They span a subset of $T\mathcal{M}_{10}$ which may be thought of as the embedding of the tangent space $T\mathcal{M}_{p+1}$ of the brane into that of space-time. Orthonormalizing the ∂_μ we obtain a new basis of $T\mathcal{M}_{p+1}$ given by some E_α . I.e. $\text{span}(E_\alpha) = T\mathcal{M}_{p+1} \subset T\mathcal{M}_{10}$. It follows from the construction that the E_α are closed under the Lie bracket, i.e. $[E_\alpha, E_\beta] \in \text{span}(E_\gamma)$. Therefore $E_\alpha^m = 0$ and the matrix E_α^μ is invertible. We may complete the set E_α to a basis of the whole tangent space, $E_A = (E_\alpha, E_a)$. Naturally, there is a dual basis of covectors, $e^A = (e^\alpha, e^a)$ which we may use as a vielbein.

Having constructed a vielbein suitable for our purposes we shall now express the DBI action in terms of that vielbein. As the two bases are dual we have

$$0 = E_\alpha e^b = E_\alpha^M e_M^b \quad (2.72)$$

Contracting with $(E_\alpha^\mu)^{-1} = e_\mu^\alpha$, we obtain

$$e_\mu^b = 0 \quad (2.73)$$

This is quite important. It means that the components e^a of the vielbein are not pulled back onto the brane world-volume whereas all the e^α are. After all, the pull-back acts as $X^*(\omega_M dx^M) = \omega_\mu d\xi^\mu$. It follows that the volume form induced onto the brane world-volume is given by the pull-back of the forms e^α

$$\sqrt{-\hat{g}_{(p+1)}}d^{p+1}\xi = \bigwedge_\alpha (X^* e^\alpha) \quad (2.74)$$

The DBI action in this frame is therefore given by

$$S_{DBI} = -T_p \int_{\mathcal{M}_{p+1}} e^{\frac{p-3}{4}\Phi} \bigwedge_\alpha (X^* e^\alpha) \quad (2.75)$$

In the final part of our discussion, we will impose some constraints on the calibration and smearing form, and show that an action of the form (1.2) can always be rewritten in the form (2.9). For the calibration form to satisfy $X^*\hat{\phi} = \sqrt{-\hat{g}_{(p+1)}}d^{p+1}\xi$, it has to include $\bigwedge_{\alpha} e^{\alpha}$. So we may assume it to be of the form $\hat{\phi} = \bigwedge_{\alpha} e^{\alpha} + \tilde{\phi}$, where $\tilde{\phi}$ is a $(p+1)$ -form which does not depend on all the indices α simultaneously and therefore includes some of the e^{α} . It follows that $X^*\tilde{\phi} = 0$. The smearing form is defined on the space transverse to the branes. This space has a one-form basis given by dy^m . As we saw above $e_{\mu}^a = 0$ and it follows that we may write the smearing form in this basis as

$$\begin{aligned}\Omega &= \frac{1}{(10-p-1)!} \Omega_{m_1 \dots m_{10-p-1}} dy^1 \wedge \dots \wedge dy^{10-p-1} \\ &= \frac{1}{(10-p-1)!} \Omega_{a_1 \dots a_{10-p-1}} e^{a_1 \dots a_{10-p-1}} = \Omega_{(p+2) \dots 9} e^{(p+2) \dots 9}\end{aligned}\quad (2.76)$$

That is, locally the smearing form is defined by a single scalar function $\Omega_{(p+2) \dots 9}$ and includes the wedge product over all the transverse components of the vielbein, $\bigwedge_a e^a$. We see immediatly that $\tilde{\phi} \wedge \Omega = 0$. Moreover

$$\hat{\phi} \wedge \Omega = e^{0 \dots 9} \Omega_{(p+2) \dots 9} \quad (2.77)$$

The trick is now to associate the indices of the function $\Omega_{(p+2) \dots 9}$ with something other than those of the relevant components of the vielbein, as we need those for the overall volume form $e^{0 \dots 9} = \sqrt{-g_{(10)}} d^{10}x$. As the form reduces to a function and we are working in flat indices, we may resolve this as follows:

$$\begin{aligned}\hat{\phi} \wedge \Omega &= e^{0 \dots 9} \Omega_{(p+2) \dots 9} = e^{0 \dots 9} \sqrt{\Omega_{(p+2) \dots 9} \Omega^{(p+2 \dots 9)}} \\ &= \sqrt{-g_{(10)}} d^{10}x |\Omega|\end{aligned}\quad (2.78)$$

with the modulus of the smearing form defined as in (2.10). As the wedge product is linear, one may immediately generalize our argument here for multiple stacks of branes, thus proving our initial assertion.

As an immediate application of the results of this section we shall take a brief look at central extensions of SUSY algebras. From the equations of motion (2.13) it follows that the smearing form is exact, $dF_{(10-p-2)} = 2\kappa_{10}^2 T_p \Omega$. Supersymmetry requires that $(e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)})$ is closed. It follows that we may write the smeared brane action (2.67) as a surface integral at infinity,

$$S_{\text{Branes}} = -\frac{1}{2\kappa_{10}^2} \int_{S_{\infty}^9} (e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)}) \wedge F_{(10-p-2)} \quad (2.79)$$

This takes the form of a charge. From the original discussion of generalized calibrated geometry in [31] we recall the fact that probe-brane actions relate to central charges in supersymmetry algebras – as one would expect for BPS objects. We conjecture that the charge defined by (2.79) has the same interpretation.

3 $\mathcal{N} = 2$ gauge-string duality in $d = 2 + 1$

Let us now apply the methods described in the previous section to the flavoring of an $\mathcal{N} = 2$ super Yang-Mills-like dual in $d = 2 + 1$. A string dual can be

found in the unflavored case by constructing a domain-wall solution in $d = 7$ gauged supergravity and then lift it to ten dimensions. It then describes a stack of NS5-branes wrapping a three-sphere. Details and physical interpretation of this solution can be found in [8] and [36]. We are first going to describe the unflavored solution using notations from [36] before studying the addition of flavors.

3.1 The unflavored solution

In the unflavored case, we consider only NS5-branes wrapping a three-sphere. So the non-zero fields in type IIB supergravity are the metric $g_{\mu\nu}$, the dilaton Φ and the NS-NS 3-form field strength H . The solution found in [36] is, in string frame

$$ds^2 = d\xi_{1,2}^2 + \frac{2z}{g^2} d\Omega_3^2 + \frac{e^{2x}}{g^2} (dz^2 + d\psi^2) + \frac{1}{g^2 \Omega} \sin^2 \psi (E_1^2 + E_2^2) \quad (3.1)$$

$$e^{2\Phi} = \left(\frac{2z}{g^2} \right)^{3/2} \frac{e^{-2A+x}}{\Omega} \quad (3.2)$$

$$\begin{aligned} H = & \frac{ge^{-2x}}{2z\Omega^{1/2}} [\cos \psi (e^{124} - e^{236} - e^{135}) - e^{2x} \sin \psi e^{127}] \\ & - \frac{ge^{-2x} \sin \psi}{\Omega^{3/2}} \left[e^{6x} \sin^2 \psi + e^{2x} (4 \cos^2 \psi + 1) - 3e^{-2x} \cos^2 \psi - \frac{\cos^2 \psi}{z} \right] e^{567} \\ & - \frac{ge^{-2x} \cos \psi}{\Omega^{3/2}} \left[e^{4x} \sin^2 \psi - 3 + e^{-4x} \cos^2 \psi - \frac{e^{2x} \sin^2 \psi}{z} \right] e^{456} \end{aligned} \quad (3.3)$$

A and x are functions of z defined as

$$e^{-2x} = \frac{I_{3/4}(z) - cK_{3/4}(z)}{I_{-1/4}(z) + cK_{1/4}(z)} \quad (3.4)$$

$$e^{A+3x/2} = z (I_{-1/4}(z) + cK_{1/4}(z)) \quad (3.5)$$

where I_α and K_α are the modified Bessel functions and c is an integration constant. In the previous equations, we used the vielbein

$$\begin{aligned} e^a &= \frac{\sqrt{2}z}{g} S^a \quad a = 1, 2, 3 & e^7 &= \frac{1}{g\Omega^{1/2}} (\cos \psi dz - e^{2x} \sin \psi d\psi) \\ e^4 &= \frac{1}{g\Omega^{1/2}} (e^{2x} \sin \psi dz + \cos \psi d\psi) & e^8 &= d\xi^1 \\ e^5 &= \frac{1}{g\Omega^{1/2}} \sin \psi E_1 & e^9 &= d\xi^2 \\ e^6 &= \frac{1}{g\Omega^{1/2}} \sin \psi E_2 & e^0 &= d\xi^0 \end{aligned} \quad (3.6)$$

with

$$\begin{aligned}
\sigma^1 &= \cos \tilde{\beta} d\tilde{\theta} + \sin \tilde{\beta} \sin \tilde{\theta} d\tilde{\phi} \\
\sigma^2 &= \sin \tilde{\beta} d\tilde{\theta} - \cos \tilde{\beta} \sin \tilde{\theta} d\tilde{\phi} \\
\sigma^3 &= d\tilde{\beta} + \cos \tilde{\theta} d\tilde{\phi} \\
S^1 &= \cos \phi \frac{\sigma^1}{2} - \sin \phi \frac{\sigma^2}{2} \\
S^2 &= \sin \theta \frac{\sigma^3}{2} - \cos \theta \left(\sin \phi \frac{\sigma^1}{2} + \cos \phi \frac{\sigma^2}{2} \right) \\
S^3 &= -\cos \theta \frac{\sigma^3}{2} - \sin \theta \left(\sin \phi \frac{\sigma^1}{2} + \cos \phi \frac{\sigma^2}{2} \right) \\
E_1 &= d\theta + \cos \phi \frac{\sigma^1}{2} - \sin \phi \frac{\sigma^2}{2} \\
E_2 &= \sin \theta \left(d\phi + \frac{\sigma^3}{2} \right) - \cos \theta \left(\sin \phi \frac{\sigma^1}{2} + \cos \phi \frac{\sigma^2}{2} \right) \\
\Omega &= e^{2x} \sin^2 \psi + e^{-2x} \cos^2 \psi \\
\theta, \tilde{\theta}, \psi &\in [0, \pi] \quad \phi, \tilde{\phi} \in [0, 2\pi[\quad \tilde{\beta} \in]0, 4\pi]
\end{aligned} \tag{3.7}$$

and $d\Omega_3^2 = \sigma^i \sigma^i$. We know that type IIB supergravity contains thirty-two supercharges that can be described by an $\mathcal{SO}(2)$ doublet of chiral spinors $\epsilon = (\epsilon^-, \epsilon^+)$. Their chirality is expressed as

$$\Gamma_{11} \epsilon = \Gamma_{1234567890} \epsilon = -\epsilon \tag{3.8}$$

This background preserves four supercharges, corresponding to $\mathcal{N} = 2$ in $d = 2 + 1$ dimensions. This means that ϵ has to verify the projections

$$\begin{aligned}
\Gamma^{1256} \epsilon &= \epsilon \\
\Gamma^{1346} \epsilon &= \epsilon \\
\Gamma^{4567} \epsilon &= \sigma_3 \epsilon
\end{aligned} \tag{3.9}$$

where σ_3 is the third Pauli matrix.

3.2 Deformation of the solution

We are now working again in Einstein frame. We first notice that, in the solution of the previous section, e^4 and e^7 are mixing the z and ψ coordinates. In order to simplify this, we make a common change of coordinates, first proposed in [40]:

$$\begin{aligned}
\rho &= \sin \psi \frac{e^{A-x/2}}{(2zg^2)^{1/4}} \\
\sigma &= \sqrt{g} \frac{\cos \psi}{(2z)^{3/4}} e^{A+3x/2}
\end{aligned} \tag{3.10}$$

We then get that $e^4 = h_1(\rho, \sigma)d\rho$ and $e^7 = h_2(\rho, \sigma)d\sigma$. Let us now deform the metric by modifying the vielbein in (3.6)

$$\begin{aligned}
e^a &= e^{-f/2} \sqrt{j(\rho, \sigma)} S^a \quad a = 1, 2, 3 & e^7 &= e^{-f/2} \sqrt{h_2(\rho, \sigma)} d\sigma \\
e^4 &= e^{-f/2} \sqrt{h_1(\rho, \sigma)} d\rho & e^8 &= e^{-f/2} d\xi^1 \\
e^5 &= e^{-f/2} \sqrt{h_1(\rho, \sigma)k(\rho, \sigma)} E_1 & e^9 &= e^{-f/2} d\xi^2 \\
e^6 &= e^{-f/2} \sqrt{h_1(\rho, \sigma)k(\rho, \sigma)} E_2 & e^0 &= e^{-f/2} d\xi^0
\end{aligned} \tag{3.11}$$

It gives us the following ansatz for the metric:

$$\begin{aligned}
ds^2 &= e^{-f(\rho, \sigma)} \left(d\xi_{1,2}^2 + j(\rho, \sigma) d\Omega_3^2 + h_1(\rho, \sigma) [d\rho^2 + k(\rho, \sigma) (E_1^2 + E_2^2)] \right. \\
&\quad \left. + h_2(\rho, \sigma) d\sigma^2 \right)
\end{aligned} \tag{3.12}$$

It is straightforward to see that this ansatz leaves the topology of the previous solution invariant.

3.3 Calibration, smearing and G-structures

We are now interested in adding flavor D5-branes to the background. Following the usual method, we first deform the unflavored solution for D5-branes. Then we find calibrated cycles where we can put supersymmetric D5-branes. We finally smear them and find a solution that includes their backreaction.

The solution in the previous section describes NS5-branes. As we are interested in the IR behaviour of the gauge dual, we want to consider D5-branes. So we first perform an S-duality on the solution. It gives a new solution of type IIB supergravity describing D5-branes, for which non-zero fields are the metric, the dilaton and the Ramond-Ramond 3-form such that

$$g_{\mu\nu}^{NS5} \rightarrow g_{\mu\nu}^{D5} \tag{3.13}$$

$$\Phi^{NS5} \rightarrow -\Phi^{D5} \tag{3.14}$$

$$H_{(3)}^{NS5} \rightarrow F_{(3)}^{D5} \tag{3.15}$$

$$\sigma_3 \rightarrow \sigma_1 \tag{3.16}$$

As we want to keep the same number of supercharges, and just deform the previous solution, we are imposing the same projections on the SUSY spinors as (3.9). We then define a new $\mathcal{SO}(2)$ doublet

$$\boldsymbol{\eta} = \begin{pmatrix} \eta^- \\ \eta^+ \end{pmatrix} = \begin{pmatrix} \epsilon^- + \epsilon^+ \\ \epsilon^- - \epsilon^+ \end{pmatrix} \tag{3.17}$$

such that (3.9) becomes

$$\begin{aligned}
\Gamma^{1256} \boldsymbol{\eta} &= \boldsymbol{\eta} \\
\Gamma^{1346} \boldsymbol{\eta} &= \boldsymbol{\eta} \\
\Gamma^{4567} \boldsymbol{\eta} &= \sigma_3 \boldsymbol{\eta}
\end{aligned} \tag{3.18}$$

Notice that $\boldsymbol{\eta}$ is still a doublet of chiral spinors that satisfies

$$\Gamma_{11} \boldsymbol{\eta} = -\boldsymbol{\eta} \tag{3.19}$$

From the third projection, we see that η^- and η^+ are both non-zero, but behave differently under the action of gamma matrices. So for each spinor we can construct a six-dimensional generalized calibration form

$$\begin{aligned}\hat{\Phi}^- &= \eta^{-T} \Gamma_{089abc} \eta^- e^{089abc} \\ \hat{\Phi}^+ &= \eta^{+T} \Gamma_{089abc} \eta^+ e^{089abc}\end{aligned}\tag{3.20}$$

Those forms can be written as

$$\begin{aligned}\hat{\Phi}^- &= e^{089} \wedge \hat{\phi}^- \\ \hat{\Phi}^+ &= e^{089} \wedge \hat{\phi}^+\end{aligned}\tag{3.21}$$

where $\hat{\phi}^+$ and $\hat{\phi}^-$ are three-forms. Using supersymmetric variations of the gravitino and the dilatino and identifying $F_{(3)}$ with a torsion term, it is possible to define two covariant derivatives $\tilde{\nabla}^+$ and $\tilde{\nabla}^-$ such that

$$\begin{aligned}\tilde{\nabla}^+ \eta^+ &= 0 \\ \tilde{\nabla}^- \eta^- &= 0\end{aligned}\tag{3.22}$$

So the existence of η^\pm imposes that the internal manifold has special holonomy, and thus admits a corresponding G -structure. With each spinor satisfying the projections (3.18), it is possible to define two different G_2 structures in the seven-dimensional space with tangent directions $\{1,2,3,4,5,6,7\}$. The corresponding associative three-forms are $\hat{\phi}^+$ and $\hat{\phi}^-$. We want the flavor branes we add to preserve the same supercharges as in the unflavored solution. From [35], we know that there is in fact an $SU(3)$ structure in that space, for which the three-dimensional calibration form is

$$\hat{\phi} = \frac{1}{2}(\hat{\phi}^- - \hat{\phi}^+)\tag{3.23}$$

So the calibration form for D5-branes in this geometry is

$$\hat{\Phi} = e^{089} \wedge \hat{\phi}\tag{3.24}$$

We have (details of the calculation can be found in Appendix B)

$$\begin{aligned}\hat{\phi}^- &= e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356} \\ \hat{\phi}^+ &= -e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}\end{aligned}\tag{3.25}$$

So,

$$\hat{\Phi} = e^{089} \wedge (e^{123} + e^{145} + e^{246} - e^{356})\tag{3.26}$$

In order to find solutions for the deformed background, we first need to provide an ansatz for the Ramond-Ramond form $F_{(3)}$:

$$\begin{aligned}F &= e^{-3\Phi/4} (F_{124}(\rho, \sigma) e^{124} + F_{135}(\rho, \sigma) e^{135} + F_{236}(\rho, \sigma) e^{236} + F_{127}(\rho, \sigma) e^{127} \\ &\quad + F_{456}(\rho, \sigma) e^{456} + F_{567}(\rho, \sigma) e^{567})\end{aligned}\tag{3.27}$$

and we assume the dilaton depends only on ρ and σ . As mentioned previously, we know from [38] that conservation of supersymmetry gives us first order differential equations that, in addition to imposing the Bianchi identity for $F_{(3)}$,

will solve the equations of motion. One way to find those equations is to study the type IIB supersymmetry transformations of the dilatino and the gravitino

$$\delta\lambda = \frac{1}{2}\Gamma^\mu\partial_\mu\Phi\boldsymbol{\eta} + \frac{1}{24}e^{\Phi/2}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\sigma_3\boldsymbol{\eta} = 0 \quad (3.28)$$

$$\delta\psi_\mu = \nabla_\mu\boldsymbol{\eta} + \frac{1}{96}e^{\Phi/2}F_{\nu\rho\sigma}(\Gamma_\mu{}^{\nu\rho\sigma} - 9\delta_\mu^\nu\Gamma^{\rho\sigma})\sigma_3\boldsymbol{\eta} = 0 \quad (3.29)$$

Another way is to use the geometric properties of the space, using G -structures and generalized calibration conditions. As stated previously, we need to assume that $\Phi = 2f$. Otherwise, we can look at the dilatino variation to get an additional condition. From it we get

$$\partial_\rho\Phi = \frac{e^{(2f-\Phi)/4}\sqrt{h_1}}{2}(F_{127} - F_{567}) \quad (3.30)$$

$$\partial_\sigma\Phi = \frac{e^{(2f-\Phi)/4}\sqrt{h_2}}{2}(F_{135} + F_{236} + F_{456} - F_{124}) \quad (3.31)$$

Then we remember that $\hat{\Phi}^-$ is a generalized calibration and $\hat{\phi}^-$ defines a G_2 structure. So we get two conditions on those forms

$$d(e^{\Phi/2}\hat{\Phi}^-) = -e^\Phi *_7 F \quad (3.32)$$

$$d(e^\Phi *_7 \hat{\phi}^-) = d(e^\Phi *_7 \hat{\Phi}^-) = 0 \quad (3.33)$$

Using the conditions on the dilaton, those two equations give us

$$f = \frac{\Phi}{2} \quad (3.34)$$

$$\partial_\rho\Phi = -\frac{j\sqrt{h_1}F_{567} + h_1\sqrt{k}}{2j} \quad (3.35)$$

$$\partial_\sigma\Phi = \frac{\sqrt{h_2}(F_{456} - 3F_{124})}{2} \quad (3.36)$$

$$\partial_\rho j = 2h_1\sqrt{k} \quad (3.37)$$

$$\partial_\sigma j = 2j\sqrt{h_2}F_{124} \quad (3.38)$$

$$\partial_\rho k = 2\sqrt{k} - \frac{h_1 k^{3/2}}{j} + k \frac{h_1^{3/2}F_{567} - \partial_\rho h_1}{h_1} \quad (3.39)$$

$$\partial_\sigma k = 0 \quad (3.40)$$

$$\partial_\rho h_2 = h_2 \frac{j\sqrt{h_1}F_{567} + h_1\sqrt{k}}{j} \quad (3.41)$$

$$\partial_\sigma h_1 = h_1\sqrt{h_2}(F_{124} - F_{456}) \quad (3.42)$$

$$F_{127} = -\frac{\sqrt{h_1 k}}{j} \quad (3.43)$$

$$F_{135} = -F_{124} \quad (3.44)$$

$$F_{236} = -F_{124} \quad (3.45)$$

Moreover, we must have

$$\begin{aligned} \partial_\rho\partial_\sigma\Phi &= \partial_\sigma\partial_\rho\Phi \\ \partial_\rho\partial_\sigma j &= \partial_\sigma\partial_\rho j \end{aligned} \quad (3.46)$$

So we get

$$\partial_\rho F_{124} = -\frac{j\sqrt{h_1}F_{124}F_{567} + h_1\sqrt{k}(3F_{124} + 2F_{456})}{2j} \quad (3.47)$$

$$\frac{\partial_\rho F_{456}}{\sqrt{h_1}} = -\frac{\partial_\sigma F_{567}}{\sqrt{h_2}} - \frac{\sqrt{h_1k}(4F_{124} + 5F_{456}) + jF_{124}F_{567}}{2j} \quad (3.48)$$

Let us now eliminate components of F in (3.35) to (3.42) and try to solve those equations. We get

$$h_1 = \frac{e^{-2\Phi}}{j} e^{a(\rho)} \quad (3.49)$$

$$h_2 = e^{-2\Phi} e^{b(\sigma)} \quad (3.50)$$

$$e^{2\Phi} = \frac{2\sqrt{k}}{j\partial_\rho j} e^a \quad (3.51)$$

$$F_{124} = \frac{e^{(a-b)/2} k^{1/4} \partial_\sigma j}{\sqrt{2\partial_\rho j} j^{3/2}} \quad (3.52)$$

$$F_{456} = \frac{e^{(a-b)/2} k^{1/4} (\partial_\sigma j \partial_\rho j - 2j \partial_\sigma \partial_\rho j)}{\sqrt{2}(j \partial_\rho j)^{3/2}} \quad (3.53)$$

$$F_{567} = \frac{\sqrt{k}(\partial_\rho j)^2 - j((2 + \sqrt{k}a') \partial_\rho j - 2\sqrt{k}\partial_\rho^2 j)}{\sqrt{2}k^{1/4} j(\partial_\rho j)^{3/2}} \quad (3.54)$$

$$\partial_\rho k = 2\sqrt{k} - ka' \quad (3.55)$$

We notice that $b(\sigma)$ is arbitrary, which corresponds to the fact that it is always possible to redefine the σ coordinate. To simplify the problem, we are taking $b = 0$ in the following sections.

3.4 Addition and smearing of flavor branes

In order to add and smear flavor branes, one needs to find the smearing form Ω . Following the prescription presented in the first part of this article, we know that this form is related to the calibration form of our background $\hat{\Phi}$ (see (3.26)) through

$$\Omega = dF = -d(e^{-\Phi} * d(e^{\Phi/2} \hat{\Phi})) \quad (3.56)$$

Using this, the ansatz for the metric and for F and the equations found in the previous section ((3.34) to (3.48)), we can deduce that the most general form of Ω is

$$\Omega = e^\Phi \left(N_{f1}(\rho, \sigma) [e^{2367} + e^{1357} - e^{1247}] + N_{f2}(\rho, \sigma) e^{4567} \right) \quad (3.57)$$

with

$$\partial_\sigma F_{124} = \sqrt{h_2} \frac{j(F_{124}F_{456} - 5F_{124}^2 + 2N_{f1}e^{2\Phi}) - 2 - 2\sqrt{h_1k}F_{567}}{2j} \quad (3.58)$$

$$\frac{\partial_\sigma F_{456}}{\sqrt{h_2}} = \frac{\partial_\rho F_{567}}{\sqrt{h_1}} + \frac{3F_{567}^2}{2} + \frac{F_{567}(4j - h_1k)}{2j\sqrt{h_1k}} + \frac{3F_{456}(F_{456} - F_{124})}{2} - e^{2\Phi} N_{f2} \quad (3.59)$$

Consistency between those equations and (3.49) to (3.55) imposes that

$$N_{f2} = N_{f1} + \frac{j}{h_1 \sqrt{k}} \partial_\rho N_{f1} \quad (3.60)$$

$$0 = 2j^2 \partial_\rho^2 j + 2e^a j \partial_\sigma^2 j + j(\partial_\rho j)^2 - e^a (\partial_\sigma j)^2 - j^2 (a' \partial_\rho j + 4e^a N_{f1}) \quad (3.61)$$

We now see that the only unknown we have is N_{f1} . Any function of ρ and σ is possible and will give first order differential equations that will solve the modified equations of motion for type IIB supergravity plus flavor embeddings. Finding a solution then consists only on solving the second-order differential equation (3.61). However, while the choice of the function N_{f1} determines which branes are smeared, we are unable to derive the embedding of the supersymmetric branes that have been smeared. One might want to recall the discussion at the end of section 2.1.3.

3.4.1 Different possibilities for the smearing form

As it was stated before, the starting point of adding smeared flavors is to choose a smearing form, which, in the case we are currently studying, corresponds to choosing a function $N_{f1}(\rho, \sigma)$.

A first possibility would be to take N_{f1} independent of ρ . It follows from (3.60) that

$$N_{f1} = N_{f2} = N_f(\sigma) \quad (3.62)$$

Then we can try to solve (3.61) by making the following ansatz for j :

$$j(\rho, \sigma) = G(\rho)^{2/3} H(\sigma)^2 \quad (3.63)$$

We obtain

$$G' = c_1 e^{a/2} \quad (3.64)$$

$$\frac{H''}{H} = N_f \quad (3.65)$$

where c_1 is a constant. In the case where $a = 0$ and N_f is a constant, we can solve this and find

$$k = (\rho + \rho_0)^2 \quad (3.66)$$

and

$$j = (c_1 \rho + c_2)^{2/3} \cos(\sqrt{-N_f} \sigma + c_3)^2 \quad \text{if } N_f \leq 0 \quad (3.67)$$

$$j = (c_1 \rho + c_2)^{2/3} \cosh(\sqrt{N_f} \sigma + c_3)^2 \quad \text{if } N_f \geq 0 \quad (3.68)$$

with c_1, c_2 and c_3 are integration constants. These provide analytic solutions to the equations of motion of type IIB supergravity with modified Bianchi identity. When looking at the dilaton behavior, we find

$$e^{2\Phi} = \frac{3(\rho + \rho_0)}{c_1(c_2 + c_1 \rho)^{1/3} \cos(c_3 + \sqrt{-N_f} \sigma)^4} \quad \text{if } N_f \leq 0 \quad (3.69)$$

$$e^{2\Phi} = \frac{3(\rho + \rho_0)}{c_1(c_2 + c_1 \rho)^{1/3} \cosh(c_3 + \sqrt{N_f} \sigma)^4} \quad \text{if } N_f \geq 0 \quad (3.70)$$

When $N_f \leq 0$, in (3.69), it is remarkable that there are singularities for $c_3 + \sqrt{-N_f}\sigma = \frac{\pi}{2} \bmod (2\pi)$. Those singularities may be a sign of the presence of the smeared flavor branes.

Another possibility would be to try to have a smearing form independent of one of the radial coordinates, instead of just the function N_{f1} as in the previous paragraph. For Ω to be independent of σ , we have to take

$$N_{f1} = \frac{N(\rho)}{\sqrt{j}} \quad (3.71)$$

Then (3.61) becomes

$$0 = 2j^2\partial_\rho^2 j + 2e^a j\partial_\sigma^2 j + j(\partial_\rho j)^2 - e^a(\partial_\sigma j)^2 - j^2 a' \partial_\rho j - 4e^a N(\rho) j^{3/2} \quad (3.72)$$

Taking here $N(\rho)$ to be constant, we get $N_{f2} = 0$ which suppresses one of the terms in the smearing form. Nevertheless, it is not obvious how to find a solution to the equation for j .

For Ω to be independent of ρ , one needs to impose k to be a constant. Then

$$a(\rho) = 2a_1\rho \quad (3.73)$$

$$N_{f1} = \frac{e^{-a_1\rho}}{\sqrt{j}} N(\sigma) \quad (3.74)$$

where a_1 is a strictly positive constant. We now have to solve:

$$0 = 2j^2\partial_\rho^2 j + 2e^{2a_1\rho} j\partial_\sigma^2 j + j(\partial_\rho j)^2 - e^{2a_1\rho}(\partial_\sigma j)^2 - 2j^2 a_1 \partial_\rho j - 4e^{a_1\rho} N(\sigma) j^{3/2} \quad (3.75)$$

In the case where $N(\sigma) = N_f$ is a constant, the smearing form is independent of any radial dependence. In that case we can find asymptotic solutions, considering ρ as the energy scale. One interesting fact is that it seems it is not possible to ignore the term involving N_f in the IR, that is when ρ goes to zero. In the IR ($\rho \rightarrow 0$), we find that

$$j = e^{2a_1\rho/3} \left(\frac{3N_f}{a_1^2 - a_1} + c_1 e^{(1-a_1)\rho} \right)^{2/3} \quad \text{if } a_1 \neq 1 \quad (3.76)$$

$$j = e^{2a_1\rho/3} (3N_f\rho + c_2 e^{-\rho})^{2/3} \quad \text{if } a_1 = 1 \quad (3.77)$$

In the UV, we have two possibilities: we can decide that the term in N_f is suppressed or plays a role. The two cases give

$$j = c_3 e^{2a_1\rho/3} \sigma^2 \quad \text{if we neglect the term in } N_f \quad (3.78)$$

$$j = e^{-2a_1\rho} \frac{N_f^2 \sigma^4}{4} \quad (3.79)$$

3.4.2 Comments on the solution

Firstly one can notice that none of the solutions presented in the previous section goes to the solution found in [36] in the limit N_{f1}, N_{f2} goes to zero, as expected from the dual gauge theory point of view.

We are trying to find a solution that describes a stack of N_c color branes plus one or several stacks of smeared flavor branes. The number of color branes is related to the Ramond-Ramond field $F_{(3)}$ through

$$\int_{S^3} F_{(3)} = 2\kappa_{10}^2 T_5 N_c \quad (3.80)$$

where S^3 is a three-sphere around the point where the color branes are placed in the four-dimensional space transverse to their world-volume. We were not able to find a constant when calculating the previous integral for the solutions of the previous section. It means that either we did not find the right transverse four-dimensional space, or these results cannot have the usual interpretation of stacks of branes.

This relates to the most prominent problem of the method presented in this section. As we mentioned in footnote 3, it is necessary to verify the existence of a cycle wrapped by the branes. As we explicitly avoided the issue of considering the embedding smeared, one cannot be certain that the above solutions do describe smeared branes. In simple cases when the smearing form does not have a term along the radial direction of the space, each component in the vielbein basis can usually be interpreted as the volume form of the space orthogonal to the brane smeared. In the case studied above, Ω has to have a term in $d\rho$. So in comparison to Klebanov-Witten, it seems that we are smearing massive flavor branes. But we were not able to determine their embedding. However, the form of Ω tells us it is not possible to smear massless flavor branes in this background. Moreover, knowing the explicit embedding of the flavor branes is not necessary to look at some properties of the gauge theory dual.

4 Conclusion

In this paper, we applied generalized calibrations and G -structures to address the problem of adding smeared flavor branes to a supergravity background. In doing so, we made a first step towards a systematic study of backgrounds with a large number of smeared flavor branes. In section 2, we showed that the smeared brane action of [34] is equivalent to those used previously in the literature on smeared flavor branes. This makes the symmetry with the Wess-Zumino term apparent and the linearity in the smearing form Ω manifest. Furthermore we were able to link the complete brane action to a conserved charge and impose strong constraints on Ω by relating it to the calibration form. While the explicit form of Ω depends on the embedding smeared, this allowed us to explain various features of the examples in section 2.1; in particular why the smearing has to preserve certain symmetries, which again implies that it is often only possible to smear several stacks of branes at once.

We exhibited the potential of our methods not only by studying known examples, yet by also flavoring a background dual to a $d = 2 + 1$, $\mathcal{N} = 2$ super Yang-Mills-like theory (See section 3). Here we found several solutions and some interesting features, notably the fact that it is not possible to smear massless flavors – a property which would be nice to understand from the point of view of the dual gauge theory.

The formalism introduced in this paper unifies the treatment of different possible embeddings for any single background, allowing for a general study

of the smearing procedure in a given background, instead of the case by case methods previously used. Even if it remains necessary to verify the existence of the cycles wrapped by the branes, their knowledge is not necessary for the actual calculation. However, as we have seen in the case of the $d = 2 + 1$, $\mathcal{N} = 2$ duality, backgrounds constructed without knowledge of the embeddings might be very difficult to interpret.

It would be interesting to work in the future on the removal of some limitations of the work we presented. Extensions of the results of this paper to type IIA backgrounds, world-volume gauge fields or the Kalb-Ramond field should be straightforward using the results of [33]. While we were able to impose strong mathematical constraints onto the smearing form Ω , we were not able to link it to the physical interpretation of a brane density. In other words, we are not providing a general way of knowing from the smearing form and the ansatz for $N_f(\rho)$ what the embeddings of the smeared flavor branes are. Even if such knowledge is not required to study some aspects of the gauge theory dual, it would give a better understanding of the way the duality is working. One might also wonder how much one can learn about the various dual gauge theories from the generic form of Ω prior to selecting one of them by making an ansatz for $N_f(\rho)$.

Finally, it might turn out to be useful to apply further relations between supersymmetry and geometry to the field of gauge/string duality. Examples of this are given by generalized complex geometry or the use of pure spinors.

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A A review of generalized calibrated geometry

We shall give a short introduction to generalized calibrated geometry [31] in relation to supersymmetric brane embeddings. The discussion given ignores the case of world-volume fluxes and follows that of the review [41].

Calibration forms and supersymmetric brane embeddings The standard method used when studying supersymmetric brane embeddings is κ -symmetry [42]. A brane embedding $X^M(\xi)$ is supersymmetric if it satisfies the equation

$$\Gamma_\kappa \epsilon = \epsilon \tag{A.1}$$

ϵ is a SUSY spinor of the background while Γ_κ is a linear map depending on the form of the embedding. For a D-brane of type IIB string theory with world-

volume gauge fields such that $\mathcal{F} = 2\pi\alpha'F - \mathcal{B} = 0$ it reduces to

$$\Gamma_\kappa = \frac{1}{(1+p)!\sqrt{-\hat{g}_{(p+1)}}} \epsilon^{\alpha_0 \dots \alpha_p} \begin{cases} (\Gamma^{11})^{\frac{p-2}{2}} \gamma_{\alpha_0 \dots \alpha_p} & \text{(IIA)} \\ \sigma_3^{\frac{p-3}{2}} \iota \sigma_2 \otimes \gamma_{\alpha_0 \dots \alpha_p} & \text{(IIB)} \end{cases} \quad (\text{A.2})$$

which is invariant under Weyl transformations and therefore valid in string and Einstein frame. The definition uses the pull-back of the space-time Γ -matrices onto the brane world-volume, $\gamma_\alpha = \partial_\alpha X^M \Gamma_M$.

Γ_κ is hermitian and squares to one. It follows that

$$\epsilon^\dagger \frac{1 - \Gamma_\kappa}{2} \epsilon = \epsilon^\dagger \frac{1 - \Gamma_\kappa}{2} \frac{1 - \Gamma_\kappa}{2} \epsilon = \left\| \frac{1 - \Gamma_\kappa}{2} \epsilon \right\|^2 \geq 0 \quad (\text{A.3})$$

Which implies that $\epsilon^\dagger \epsilon \geq \epsilon^\dagger \Gamma_\kappa \epsilon$ with equality if and only if the embedding is supersymmetric. Normalizing the spinor such that $\epsilon^\dagger \epsilon = 1$ and using (A.2), we may rephrase this as

$$\sqrt{-\hat{g}_{(p+1)}} \geq \frac{1}{(p+1)!} \epsilon^{\alpha_0 \dots \alpha_p} \begin{cases} \epsilon^\dagger (\Gamma^{11})^{\frac{p-2}{2}} \gamma_{\alpha_0 \dots \alpha_p} \epsilon & \text{(IIA)} \\ \epsilon^\dagger \sigma_3^{\frac{p-3}{2}} \iota \sigma_2 \otimes \gamma_{\alpha_0 \dots \alpha_p} \epsilon & \text{(IIB)} \end{cases} \quad (\text{A.4})$$

Equality holds if and only if the embedding is supersymmetric. Now the right hand side of (A.4) may be written as the pull-back of a differential form defined in space-time.

$$\hat{\phi} = \frac{1}{(p+1)!} e^{a_0 \dots a_p} \begin{cases} \epsilon^\dagger (\Gamma^{11})^{\frac{p-2}{2}} \Gamma_{a_0 \dots a_p} \epsilon & \text{(IIA)} \\ \epsilon^\dagger \sigma_3^{\frac{p-3}{2}} \iota \sigma_2 \otimes \Gamma_{a_0 \dots a_p} \epsilon & \text{(IIB)} \end{cases} \quad (\text{A.5})$$

$\hat{\phi}$ is known as the calibration form. A criterion for supersymmetry of an embedding that is alternative to (A.1) is then given by the following

$$X^* \hat{\phi} = \sqrt{-\hat{g}_{(p+1)}} d^{p+1} \xi \quad (\text{A.6})$$

that is, the pull-back of the calibration form onto the world-volume is equal to the induced volume form.

One may obtain $\hat{\phi}$ directly from its definition (A.5) and the knowledge of the projections imposed onto the SUSY spinors. We shall give an example of this in appendix B.

A more formal definition Formally one defines a calibration on a Riemannian manifold as a $(p+1)$ -form $\hat{\phi}$ satisfying

$$d\hat{\phi} = 0 \quad \hat{\phi}|_{\xi^{p+1}} \leq \eta_{(p+1)}|_{\xi^{p+1}} \quad (\text{A.7})$$

Here ξ^p is a set of vectors specifying a tangent $(p+1)$ -plane to a $(p+1)$ -cycle Σ_{p+1} while $\eta_{(p+1)} = \sqrt{-\hat{g}_{(p+1)}} d^{p+1} \xi$ is the volume form induced onto that cycle. The cycle Σ_{p+1} is calibrated if the above bound is saturated, i.e. if $\hat{\phi}|_{\xi^{p+1}} = \eta_{(p+1)}|_{\xi^{p+1}}$.

As we have seen above (A.6), κ -symmetric brane embeddings satisfy the volume bound, which can be thought of as a BPS-bound. In this and in the next

paragraph we shall turn to the issue of the closure of (A.5). For a background without fluxes, the issue is rather easily resolved. From the gravitino variation

$$\delta_\epsilon \psi_M = D_M \epsilon = 0 \quad (\text{A.8})$$

it follows that the SUSY spinor ϵ is covariantly constant. As the covariant derivative of both the vielbein and the tangent-space Γ -matrices does also vanish it follows that

$$d\hat{\phi} = \nabla \wedge \hat{\phi} = 0 \quad (\text{A.9})$$

$\nabla \wedge \hat{\phi}$ is to be thought of as a formal expression. The wedge product antisymmetrizes over the relevant indices and, as the Levi-Civita connection is symmetric in two of its indices, it follows that the first equality holds. As all the ingredients of (A.5) are covariantly constant, it follows that the exterior derivative is closed.

There is a nice interpretation of the closure of the calibration form. Let us assume that we deform the calibrated cycle Σ_{p+1} to Σ'_{p+1} . The two cycles differ by a boundary $\Sigma_{p+1} - \Sigma'_{p+1} = \delta \Xi_{p+2}$. More formally we would not consider Σ'_{p+1} as a deformation, yet as a cycle within the homology class defined by Σ_{p+1} . We use Stokes theorem to establish

$$\text{Vol}(\Sigma_{p+1}) = \int_{\Sigma_{p+1}} \hat{\phi} = \int_{\Xi_{p+2}} d\hat{\phi} + \int_{\Sigma'_{p+1}} \hat{\phi} = \int_{\Sigma'_{p+1}} \hat{\phi} \leq \text{Vol}(\Sigma'_{p+1}) \quad (\text{A.10})$$

The final inequality uses (A.4). It follows that the calibrated cycle Σ_{p+1} is a minimal volume cycle. This matches nicely with our experience from string theory. In the absence of fluxes branes wrap minimal volume cycles.

Generalized calibrations The κ -symmetry matrix (A.2) does not change in the presence of Ramond-Ramond background fields and thus neither does the definition of the calibration form or the supersymmetry condition (A.6). Background fluxes however deform branes such that they do not longer wrap minimal volume cycles. For a background with fluxes we do therefore not expect the calibration form (A.5) to be closed. Rather, it's exterior differential should be related to the flux. Indeed, in all the examples studied in section 2.1 the calibration satisfied

$$d(e^{\frac{p-3}{4}\Phi} \hat{\phi}) = F_{(p+2)} \quad (\text{A.11})$$

In this case, one speaks of a generalized calibration, a concept which was first introduced in [31].

There are several ways to prove (A.11). For all the examples of 2.1, the equality held after we imposed the BPS equations, so it should be no surprise that (A.11) is intimately linked to the supersymmetry of the background. The original proof [31] showed that the expression $(e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)})$ appears as the central charge of a supersymmetry algebra and must therefore be topological and thus exact. It is also possible to verify (A.11) in terms of the dilatino and gravitino supersymmetry transformations.

Before doing so however, we shall take a look at the appropriate generalization of (A.10). To do so we shall assume that both the brane and the background fields are static. It follows that the energy of the system is proportional to its action – with the proportionality constant being infinity. Moreover, minimum

energy configurations will therefore minimize the brane action. Let Σ_{p+1} be the supersymmetric cycle wrapped by the brane and $\Sigma'_{p+1} = \Sigma_{p+1} + \delta\Xi_{(p+2)}$ a deformation. Then (setting $T_p = 1$)

$$\begin{aligned}
\Delta E &\propto S_{\Sigma'_{p+1}} - S_{\Sigma_{p+1}} \\
&= \int_{\Sigma'_{p+1}} (e^{\frac{p-3}{4}\Phi} \eta - C_{(p+1)}) - \int_{\Sigma_{p+1}} (e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)}) \\
&\geq \int_{\delta\Xi_{p+2}} (e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)}) \\
&= \int_{\Xi_{p+2}} d(e^{\frac{p-3}{4}\Phi} \hat{\phi} - C_{(p+1)}) = 0
\end{aligned} \tag{A.12}$$

The inequality in the second line used again (A.4). It follows that supersymmetric, static embeddings are minimum energy configurations.

B Finding the calibration form – an explicit example

As an example we will calculate the calibration form for the theory of section 3. Apart from the definition (A.5) we will need the projections imposed on the background SUSY spinors. To simplify things we perform a change of basis on the spinors taking $\sigma_1 \mapsto \sigma_3$. As a result of this transformations, the two Majorana-Weyl spinors in $\zeta = \begin{pmatrix} \zeta^- \\ \zeta^+ \end{pmatrix}$ decouple

$$\Gamma^{1256}\zeta^\mp = \zeta^\mp \quad \Gamma^{1346}\zeta^\mp = \zeta^\mp \quad \Gamma^{4567}\zeta^\mp = \pm\zeta^\mp \tag{B.1}$$

We will also need the fact that IIB supergravity is chiral, with the chirality chosen such that

$$\Gamma_{11}\zeta^\mp = \Gamma_{123\dots 890}\zeta^\mp = -\zeta^\mp \tag{B.2}$$

Note that our change of basis does also affect the definition of the calibration form (A.5) – we obtain two calibration forms, $\hat{\phi}^\mp$. Note also that we will work in flat indices.

Before looking at the most generic case, we shall look at a few examples of how to calculate components of $\hat{\phi}^\mp$

$$\begin{aligned}
\hat{\phi}_{089123}^\mp &= \zeta^{\mp T} \Gamma_{089123} \zeta^\mp = \zeta^{\mp T} \Gamma_{4567} \zeta^\mp = \pm 1 \\
\hat{\phi}_{089145}^\mp &= \zeta^{\mp T} \Gamma_{2367} \zeta^\mp = \zeta^{\mp T} \Gamma_{4567} \zeta^\mp = \pm 1 \\
\hat{\phi}_{089167}^\mp &= \zeta^{\mp T} \Gamma_{2345} \zeta^\mp = 1
\end{aligned} \tag{B.3}$$

These examples show nicely that the two forms disagree on those cycles making use of the Γ^{4567} projection. As the two forms need to disagree by an overall sign for a cycle to be supersymmetric, it follows that cycles involving the 7 direction cannot be supersymmetric. One can arrive at the same result directly from the κ -symmetry condition.

The more difficult step is to show why components such as

$$\hat{\phi}_{089567}^\mp = \zeta^{\mp T} \Gamma_{1234} \zeta^\mp = \zeta^{\mp T} \Gamma_{26} \zeta^\mp \tag{B.4}$$

vanish. Starting from the projections

$$\frac{1 - \Gamma^{1346}}{2} \zeta^\mp = 0 \quad \frac{1 - \Gamma^{1256}}{2} \zeta^\mp = 0 \quad \frac{1 \mp \Gamma^{4567}}{2} \zeta^\mp = 0 \quad (\text{B.5})$$

we define orthogonal projectors

$$\frac{1 + \Gamma^{1346}}{2} \quad \frac{1 + \Gamma^{1256}}{2} \quad \frac{1 \pm \Gamma^{4567}}{2} \quad (\text{B.6})$$

which may be used to project an arbitrary spinor ψ onto the subspace of spinors satisfying (B.5) because

$$\left(\frac{1 - \Gamma^{1346}}{2} \right) \left(\frac{1 + \Gamma^{1346}}{2} \right) \psi = 0 \quad (\text{B.7})$$

independently of the choice of ψ . This is simply the defining property of orthogonal projections. Note that ζ may be assumed to be invariant under the orthogonal projections, as it satisfies (B.5). Applying this to the question of $\hat{\phi}_{089567}^\mp$,

$$\begin{aligned} \hat{\phi}_{089567}^\mp &= \zeta^{\mp T} \Gamma_{26} \zeta^\mp \\ &= \frac{1}{4} [(1 + \Gamma^{1346}) \zeta]^T \Gamma^{26} (1 + \Gamma^{1346}) \zeta \\ &= \frac{1}{4} \zeta^{\mp T} (\Gamma^{26} - \Gamma^{1234} + \Gamma^{1234} - \Gamma^{26}) \zeta^\mp \\ &= 0 \end{aligned} \quad (\text{B.8})$$

Note however that it does not appear to be obvious which of the projections (B.6) one has to choose to show that a particular component of $\hat{\phi}$ vanishes. To give an example of this, let's look at

$$\begin{aligned} \hat{\phi}_{089124}^\mp &= \pm \zeta^{\mp T} \Gamma_{34} \zeta^\mp \\ &= \pm \frac{1}{4} \zeta^{\mp T} (1 + \Gamma^{1346}) \Gamma^{34} (1 + \Gamma^{1346}) \zeta^\mp \\ &= \pm \frac{1}{4} \zeta^{\mp T} (\Gamma^{34} - \Gamma^{16} - \Gamma^{16} + \Gamma^{34}) \zeta^\mp \\ \hat{\phi}_{089124}^\mp &= \pm \frac{1}{4} \zeta^{\mp T} (1 \pm \Gamma^{4567}) \Gamma^{34} (1 \pm \Gamma^{4567}) \zeta^\mp \\ &= \frac{1}{4} \zeta^{\mp T} (\pm \Gamma^{34} - \Gamma^{3567}) (1 \pm \Gamma^{4567}) \zeta^\mp \\ &= \frac{1}{4} \zeta^{\mp T} (\pm \Gamma^{34} - \Gamma^{3567} + \Gamma^{3567} \mp \Gamma^{34}) \zeta^\mp = 0 \end{aligned} \quad (\text{B.9})$$

Let's try to look at a generic case. There is no summation in the following. Instead the indices $(a, b, c, d, e, f, g) \in \{1, \dots, 7\}$ are all independent and mutually non-equal, $a \neq b, a \neq c, \dots, f \neq g$.

$$\begin{aligned} \pm 4 \hat{\phi}_{089abc}^\mp &= \zeta^{\mp T} (1 \pm \Gamma^{cdef}) \Gamma^{defg} (1 \pm \Gamma^{cdef}) \zeta^\mp \\ &= \zeta^{\mp T} (\Gamma^{defg} \mp \Gamma^{cg}) (1 \pm \Gamma^{cdef}) \zeta^\mp \\ &= \zeta^{\mp T} (\Gamma^{defg} \mp \Gamma^{cg} \pm \Gamma^{cg} - \Gamma^{defg}) \zeta^\mp = 0 \end{aligned} \quad (\text{B.10})$$

In the first line we used the chirality matrix Γ_{11} to change Γ_{089abc} into Γ^{defg} . In the process we might have picked up an overall minus sign, which we moved

together with the factor 4 to the left hand side. In the projection matrices we have Γ -matrices assumed to be of the form Γ^{cdef} . Here there is again a sign ambiguity, as we have moved c to the left and as the projection might involving Γ^7 . Note that we have the same sign in both parentheses, so in the following lines we will always have either the upper signs or the lower signs, never a mixture of the two – which is why $\pm\mp = -$ in the second to last equality. Similarly we shall now take a look at

$$\begin{aligned}\pm 4\hat{\phi}_{089abc}^{\mp} &= \zeta^{\mp T} (1 \pm \Gamma^{abcd}) \Gamma^{defg} (1 \pm \Gamma^{abcd}) \zeta^{\mp} \\ &= \zeta^{\mp T} (\Gamma^{defg} \pm \Gamma^{abcefg} \mp \Gamma^{abcefg} - \Gamma^{defg}) \zeta^{\mp} = 0\end{aligned}\quad (\text{B.11})$$

Equation (B.10) is a very potent result. It follows immediately that

$$\begin{aligned}\hat{\phi}_{124} &= 0 & \hat{\phi}_{125} &= 0 & \hat{\phi}_{126} &= 0 & \hat{\phi}_{127} &= 0 & \hat{\phi}_{134} &= 0 & \hat{\phi}_{135} &= 0 \\ \hat{\phi}_{136} &= 0 & \hat{\phi}_{137} &= 0 & \hat{\phi}_{147} &= 0 & \hat{\phi}_{157} &= 0 & \hat{\phi}_{234} &= 0 & \hat{\phi}_{235} &= 0 \\ \hat{\phi}_{236} &= 0 & \hat{\phi}_{237} &= 0 & \hat{\phi}_{245} &= 0 & \hat{\phi}_{247} &= 0 & \hat{\phi}_{256} &= 0 & \hat{\phi}_{267} &= 0 \\ \hat{\phi}_{345} &= 0 & \hat{\phi}_{346} &= 0 & \hat{\phi}_{357} &= 0 & \hat{\phi}_{367} &= 0 & \hat{\phi}_{457} &= 0 & \hat{\phi}_{467} &= 0\end{aligned}\quad (\text{B.12})$$

Similarly we gather from (B.11)

$$\hat{\phi}_{146} = 0 \quad \hat{\phi}_{156} = 0 \quad \hat{\phi}_{456} = 0 \quad (\text{B.13})$$

All these things considered we are able to reproduce the two calibration forms exhibited in [36],

$$\begin{aligned}\hat{\phi}^- &= e^{089} \wedge (e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}) \\ \hat{\phi}^+ &= e^{089} \wedge (-e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356})\end{aligned}\quad (\text{B.14})$$

There is a second result following immediately from equations (B.10) and (B.11). For the SUSY projections not to be mutually exclusive they have to have pairwise two indices in common. Note that this may be easily generalized to arbitrary dimensions. In general one finds that if the SUSY projections take the form of antisymmetrized Gamma matrices with four indices, $\Gamma^{abcd}\zeta = \zeta$, different projections have to have an even number of indices in common (zero or two; four means that the projections are equal) in order to be compatible. Compatible means that this requirement is necessary for a spinor ζ satisfying all projections to exist. This result simply requires the properties of the Dirac algebra.

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